12. INDEFINITE INTEGRAL

INTEGRATION *It is the inverse process of differentiation.*

If the derivative of $F(x)$ is $f(x)$ then we say that the *antiderivative* or *integral* of $f(x)$ is $F(x)$ and we write,

$$\int f(x)\,dx = F(x).$$

Thus, $$\frac{d}{dx}[F(x)] = f(x) \;\Rightarrow\; \int f(x)\;dx = F(x).$$

Example Since $\frac{d}{dx}(\sin x) = \cos x$, we have $\int \cos x\,dx = \sin x.$

Moreover, if C is any constant then $\frac{d}{dx}(\sin x + C) = \cos x.$

So, in general, $\int \cos x\,dx = (\sin x + C).$

Clearly, different values of C will give different integrals.

Thus, a given function may have an indefinite number of integrals. Because of this property, we call these integrals *indefinite integrals.*

Thus, $\frac{d}{dx}[F(x)] = f(x) \;\Rightarrow\; \int f(x)\,dx = F(x) + C,$ where C is a constant, called the *constant of integration.* Any function to be integrated is known as an *integrand.*

The following two results are a direct consequence of the definition of an integral.

RESULT 1 $\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + C,\ when\ n \neq -1.$

PROOF We have, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{(n+1)} = x^n.$

$\therefore$ $$\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + C.$$

Thus, we have

(i) $\int x^6 dx = \frac{x^{(6+1)}}{(6+1)} + C = \frac{x^7}{7} + C.$

(ii) $\int x^{2/3}dx = \dfrac{x^{\left(\frac{2}{3}+1\right)}}{\left(\frac{2}{3}+1\right)} + C = \dfrac{3}{5}x^{5/3} + C.$

(iii) $\int x^{-3/4}dx = \dfrac{x^{\left(-\frac{3}{4}+1\right)}}{\left(-\frac{3}{4}+1\right)} = 4x^{1/4} + C.$

RESULT 2 $\int \dfrac{1}{x}dx = \log|x| + C, \quad$ *where* $x \neq 0.$

PROOF Clearly, either $x > 0$ or $x < 0.$

Case I *When* $x > 0$

In this case, $|x| = x.$

$\therefore \quad \dfrac{d}{dx}[\log|x|] = \dfrac{d}{dx}(\log x) = \dfrac{1}{x}.$

So, we have, $\int \dfrac{1}{x}dx = \log|x| + C.$

Case II *When* $x < 0$

In this case $|x| = -x.$

$\therefore \quad \dfrac{d}{dx}[\log|x|] = \dfrac{d}{dx}[\log(-x)] = \dfrac{1}{(-x)}\cdot(-1) = \dfrac{1}{x}\cdot$

So, we have $\int \dfrac{1}{x}\,dx = \log|x| + C.$

Thus, from both the cases, we have $\int \dfrac{1}{x}\,dx = \log|x| + C.$

FORMULAE

On the basis of differentiation and the definition of integration, we have the following results.

1. $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \Rightarrow \int x^n dx = \dfrac{x^{n+1}}{(n+1)} + C$

2. $\dfrac{d}{dx}(\log|x|) = \dfrac{1}{x} \Rightarrow \int \dfrac{1}{x}\,dx = \log|x| + C$

3. $\dfrac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + C$

4. $\dfrac{d}{dx}\left(\dfrac{a^x}{\log a}\right) = a^x \Rightarrow \int a^x dx = \dfrac{a^x}{\log a} + C$

5. $\dfrac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$

6. $\dfrac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x\,dx = -\cos x + C$

7. $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\, dx = \tan x + C$

8. $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x\, dx = -\cot x + C$

9. $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x\, dx = \sec x + C$

10. $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x \Rightarrow \int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$

12. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{(1+x^2)} \Rightarrow \int \frac{1}{(1+x^2)}\, dx = \tan^{-1} x + C$

13. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$

With the help of the above formulae, it is easy to evaluate the following integrals.

EXAMPLE 1 *Evaluate:*

(i) $\int x^9 dx$ *(ii)* $\int \sqrt[3]{x}\, dx$ *(iii)* $\int dx$

(iv) $\int \frac{1}{x^2}\, dx$ *(v)* $\int \frac{1}{x^{1/3}}\, dx$ *(vi)* $\int 5^x\, dx$

SOLUTION Using the standard formulae, we have

(i) $\int x^9 dx = \frac{x^{(9+1)}}{(9+1)} + C = \frac{x^{10}}{10} + C.$

(ii) $\int \sqrt[3]{x}\, dx = \int x^{1/3} dx = \frac{x^{\left(\frac{1}{3}+1\right)}}{\left(\frac{1}{3}+1\right)} + C = \frac{3}{4} x^{4/3} + C.$

(iii) $\int dx = \int x^0 dx = \frac{x^{(0+1)}}{(0+1)} + C = x + C.$

(iv) $\int \frac{1}{x^2}\, dx = \int x^{-2} dx = \frac{x^{(-2+1)}}{(-2+1)} + C = -\frac{1}{x} + C.$

(v) $\int \frac{1}{x^{1/3}} dx = \int x^{-1/3} dx = \frac{x^{\left(-\frac{1}{3}+1\right)}}{\left(-\frac{1}{3}+1\right)} + C = \frac{3}{2} x^{2/3} + C.$

(vi) $\int 5^x\, dx = \frac{5^x}{\log 5} + C.$

Some Standard Results on Integration

THEOREM 1 $\dfrac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x).$

PROOF Let $\int f(x)\,dx = F(x).$ … (i)

Then, $\dfrac{d}{dx}\{F(x)\} = f(x)$ [by def. of integral].

$\therefore \quad \dfrac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x)$ [using (i)].

THEOREM 2 $\int k \cdot f(x)\ dx = k \cdot \int f(x)\ dx,$ *where k is a constant.*

PROOF Let $\int f(x)\ dx = F(x).$ … (i)

Then, $\dfrac{d}{dx}\{F(x)\} = f(x).$ … (ii)

$\therefore \quad \dfrac{d}{dx}\{k \cdot F(x)\} = k \cdot \dfrac{d}{dx}\{F(x)\} = k \cdot f(x)$ [using (ii)].

So, by the definition of an integral, we have

$\int \{k \cdot f(x)\}\ dx = k \cdot F(x) = k \cdot \int f(x)\ dx$ [using (i)].

EXAMPLE 2 *Evaluate:*

(i) $\int 3x^2 dx$ *(ii)* $\int 2^{(x+3)} dx$

SOLUTION (i) $\int 3x^2 dx = 3\int x^2 dx = 3 \cdot \dfrac{x^3}{3} + C = x^3 + C.$

(ii) $\int 2^{(x+3)} dx = \int 2^x \cdot 2^3 dx = 8\int 2^x\,dx = 8 \cdot \dfrac{2^x}{\log 2} + C = \dfrac{2^{(x+3)}}{\log 2} + C.$

THEOREM 3 *(i)* $\int \{f_1(x) + f_2(x)\}\ dx = \int f_1(x)\ dx + \int f_2(x)\ dx$

(ii) $\int \{f_1(x) - f_2(x)\}\ dx = \int f_1(x)\ dx - \int f_2(x)\ dx$

PROOF (i) Let $\int f_1(x)\ dx = F_1(x)$ and $\int f_2(x)\ dx = F_2(x).$ … (i)

Then, $\dfrac{d}{dx}\{F_1(x)\} = f_1(x)$ and $\dfrac{d}{dx}\{F_2(x)\} = f_2(x).$ … (ii)

Now, $\dfrac{d}{dx}\{F_1(x) + F_2(x)\} = \dfrac{d}{dx}\{F_1(x)\} + \dfrac{d}{dx}\{F_2(x)\}$

$= f_1(x) + f_2(x)$ [using (ii)].

$\therefore \quad \int \{f_1(x) + f_2(x)\}\ dx = F_1(x) + F_2(x)$

$= \int f_1(x)\ dx + \int f_2(x)\ dx$ [using (i)].

Similarly, (ii) can be proved.

REMARK In general, we have

$\int \{k_1 \cdot f_1(x) \pm k_2 \cdot f_2(x) \pm \ldots \pm k_n \cdot f_n(x)\}\ dx$

$= k_1 \cdot \int f_1(x)\,dx \pm k_2 \cdot \int f_2(x)\,dx \pm \ldots \pm k_n \cdot \int f_n(x)\,dx.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int\left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

(ii) $\int(3\sin x-4\cos x+5\sec^2 x-2\operatorname{cosec}^2 x)\,dx$

(iii) $\int(1-x)(2+3x)(5-4x)\,dx$

(iv) $\int\left(\frac{3x^4-5x^3+4x^2-x+2}{x^3}\right)dx$ *(v)* $\int\left(x^2+\frac{1}{x^2}\right)^3 dx$

SOLUTION (i) $\int\left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

$$=5\int x^3dx+2\int x^{-5}dx-7\int x\,dx+\int x^{-1/2}dx+5\int\frac{1}{x}\,dx$$

$$=5\cdot\frac{x^4}{4}+2\cdot\frac{x^{-4}}{(-4)}-7\cdot\frac{x^2}{2}+\frac{x^{1/2}}{(1/2)}+5\log|x|+C$$

$$=\frac{5x^4}{4}-\frac{1}{2x^4}-\frac{7x^2}{2}+2\sqrt{x}+5\log|x|+C.$$

(ii) $\int(3\sin x-4\cos x+5\sec^2 x-2\operatorname{cosec}^2 x)\,dx$

$$=3\int\sin x\,dx-4\int\cos x\,dx+5\int\sec^2 x\,dx-2\int\operatorname{cosec}^2 x\,dx$$

$$=3(-\cos x)-4\sin x+5\tan x-2(-\cot x)+C$$

$$=(-3\cos x-4\sin x+5\tan x+2\cot x+C).$$

(iii) $\int(1-x)(2+3x)(5-4x)\,dx=\int(10-3x-19x^2+12x^3)\,dx$

$$=10\int dx-3\int x\,dx-19\int x^2dx+12\int x^3dx$$

$$=10x-3\cdot\frac{x^2}{2}-19\cdot\frac{x^3}{3}+12\cdot\frac{x^4}{4}+C$$

$$=10x-\frac{3x^2}{2}-\frac{19x^3}{3}+3x^4+C.$$

(iv) $\int\left(\frac{3x^4-5x^3+4x^2-x+2}{x^3}\right)dx=\int\left(3x-5+\frac{4}{x}-\frac{1}{x^2}+\frac{2}{x^3}\right)dx$

[dividing each term by x^3]

$$=3\int x\,dx-5\int dx+4\int\frac{1}{x}\,dx-\int x^{-2}dx+2\int x^{-3}dx$$

$$=3\cdot\frac{x^2}{2}-5x+4\log|x|-\left(-\frac{1}{x}\right)+2\left(\frac{x^{-2}}{-2}\right)+C$$

$$=\frac{3x^2}{2}-5x+4\log|x|+\frac{1}{x}-\frac{1}{x^2}+C.$$

(v) $\int\left(x^2+\frac{1}{x^2}\right)^3 dx = \int\left(x^6+\frac{1}{x^6}+3x^2+\frac{3}{x^2}\right)dx$

$= \int x^6 dx + \int x^{-6} dx + 3\int x^2 dx + 3\int \frac{1}{x^2}\, dx$

$= \frac{x^7}{7} + \frac{x^{-5}}{(-5)} + 3\cdot\frac{x^3}{3} + 3\cdot\left(-\frac{1}{x}\right) + C$

$= \frac{x^7}{7} - \frac{1}{5x^5} + x^3 - \frac{3}{x} + C.$

EXAMPLE 2 *Evaluate: (i)* $\int \frac{(x^3+4x^2-3x-2)}{(x+2)}\, dx$ *(ii)* $\int\left(\frac{x^4+1}{x^2+1}\right)dx$ **[CBSE 2011]**

SOLUTION (i) On dividing (x^3+4x^2-3x-2) by $(x+2)$, we get

$$\int \frac{(x^3+4x^2-3x-2)}{(x+2)}dx = \int\left\{x^2+2x-7+\frac{12}{x+2}\right\}dx$$

$$= \int x^2 dx + 2\int x\, dx - 7\int dx + 12\int \frac{1}{x+2}\, dx$$

$$= \frac{x^3}{3} + 2\cdot\frac{x^2}{2} - 7x + 12\log|x+2| + C$$

$$= \frac{x^3}{3} + x^2 - 7x + 12\log|x+2| + C.$$

(ii) On dividing (x^4+1) by (x^2+1), we get

$$\int\left(\frac{x^4+1}{x^2+1}\right)dx = \int\left[x^2-1+\frac{2}{(x^2+1)}\right]dx$$

$$= \int x^2 dx - \int dx + 2\int \frac{1}{x^2+1}\, dx$$

$$= \frac{x^3}{3} - x + 2\tan^{-1}x + C.$$

EXAMPLE 3 *Evaluate:*

(i) $\int \tan^2 x\, dx$ *(ii)* $\int \cot^2 x\, dx$ *(iii)* $\int \sin^2\frac{x}{2}\, dx$

SOLUTION (i) $\int \tan^2 x\, dx = \int(\sec^2 x - 1)\, dx$

$$= \int \sec^2 x\, dx - \int dx = \tan x - x + C.$$

(ii) $\int \cot^2 x\, dx = \int(\operatorname{cosec}^2 x - 1)\, dx$

$$= \int \operatorname{cosec}^2 x\, dx - \int dx = -\cot x - x + C.$$

(iii) We know that $2\sin^2\frac{x}{2} = (1-\cos x)$.

$$\therefore \quad \int \sin^2 \frac{x}{2}\, dx = \frac{1}{2}\int (1-\cos x)\, dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos x\, dx\right] = \frac{1}{2}x - \frac{1}{2}\sin x + C.$$

EXAMPLE 4 *Evaluate* $\int \sqrt{1-\sin 2x}\, dx.$ **[CBSE 2004]**

SOLUTION $\int \sqrt{1-\sin 2x}\, dx = \int (\cos^2 x + \sin^2 x - 2\sin x \cos x)^{1/2} dx$

$$= \int \sqrt{(\cos x - \sin x)^2}\, dx$$

$$= \int (\cos x - \sin x)\, dx = \int \cos x\, dx - \int \sin x\, dx$$

$$= \sin x - (-\cos x) + C = \sin x + \cos x + C.$$

EXAMPLE 5 *Evaluate:*

(i) $\int \frac{dx}{1+\sin x}$ **[CBSE 2002C]** *(ii)* $\int \left(\frac{\sin x}{1+\sin x}\right) dx$

SOLUTION (i) $\int \frac{dx}{(1+\sin x)} = \int \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)}\, dx$

$$= \int \frac{(1-\sin x)}{(1-\sin^2 x)}\, dx = \int \frac{(1-\sin x)}{\cos^2 x}\, dx$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx = \int (\sec^2 x - \sec x \tan x)\, dx$$

$$= \int \sec^2 x\, dx - \int \sec x \tan x\, dx = \tan x - \sec x + C.$$

(ii) $\int \left(\frac{\sin x}{1+\sin x}\right) dx = \int \frac{(1+\sin x) - 1}{(1+\sin x)} dx$

$$= \int \left(1 - \frac{1}{1+\sin x}\right) dx = \int dx - \int \frac{1}{(1+\sin x)} dx$$

$$= \int dx - \int \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)}\, dx$$

$$= \int dx - \int \frac{(1-\sin x)}{\cos^2 x} dx = \int dx - \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int dx - \int \sec^2 x\, dx + \int \sec x \tan x\, dx = x - \tan x + \sec x + C.$$

EXAMPLE 6 *Evaluate* $\int \frac{\sec x}{(\sec x + \tan x)}\, dx.$

SOLUTION $\int \frac{\sec x}{(\sec x + \tan x)}\, dx = \int \frac{\sec x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)}\, dx$

$$= \int \frac{(\sec^2 x - \sec x \tan x)}{(\sec^2 x - \tan^2 x)}\, dx$$

$$= \int (\sec^2 x - \sec x \tan x)\, dx$$

$$= \int \sec^2 x\, dx - \int \sec x \tan x\, dx = \tan x - \sec x + C.$$

EXAMPLE 7 *Evaluate:*

(i) $\int \left(\frac{4 - 5\cos x}{\sin^2 x} \right) dx$ *(ii)* $\int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx$

(iii) $\int \frac{1}{\sin^2 x \cos^2 x}\, dx$ **[CBSE 2014]** *(iv)* $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\, dx$

SOLUTION (i) $\int \left(\frac{4 - 5\cos x}{\sin^2 x} \right) dx = \int \left(\frac{4}{\sin^2 x} - \frac{5\cos x}{\sin^2 x} \right) dx$

$$= \int (4\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x)\, dx$$

$$= 4\int \operatorname{cosec}^2 x\, dx - 5\int \operatorname{cosec} x \cot x\, dx$$

$$= 4(-\cot x) - 5(-\operatorname{cosec} x) + C$$

$$= -4\cot x + 5\operatorname{cosec} x + C.$$

(ii) $\int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx = \int \frac{2\sin^2 x}{2\cos^2 x} dx = \int \tan^2 x\, dx$

$$= \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int dx$$

$$= \tan x - x + C.$$

(iii) $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \right) dx$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx = \tan x - \cot x + C.$$

(iv) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx = \int \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x} \right) dx$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx = -\cot x - \tan x + C.$$

EXAMPLE 8 *Evaluate* $\int \left(\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} \right) dx.$ **[CBSE 2013]**

SOLUTION $\int \left(\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} \right) dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{(\cos x - \cos \alpha)} dx$

$$= 2\int \frac{(\cos^2 x - \cos^2 \alpha)}{(\cos x - \cos \alpha)}\, dx = 2\int (\cos x + \cos \alpha)\, dx$$

$$= 2\int \cos x\, dx + 2\cos \alpha \cdot \int dx = 2\sin x + 2x\cos \alpha + C.$$

EXAMPLE 9 *Evaluate* $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx.$ **[CBSE 2003]**

SOLUTION $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx = \int \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right\} dx$

$$= \int \tan^{-1}(\tan x)\, dx = \int x\, dx = \frac{x^2}{2} + C.$$

EXAMPLE 10 *Evaluate* $\int \sin^{-1}(\cos x)\, dx.$

SOLUTION $\int \sin^{-1}(\cos x)\, dx = \int \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} dx$

$$= \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2} \cdot \int dx - \int x\, dx = \frac{\pi x}{2} - \frac{x^2}{2} + C.$$

EXAMPLE 11 *Evaluate* $\int \tan^{-1}(\sec x + \tan x)\, dx.$

SOLUTION $\int \tan^{-1}(\sec x + \tan x)\, dx = \int \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) dx$

$$= \int \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) dx = \int \tan^{-1}\left\{\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right\} dx$$

$$= \int \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\} dx$$

$$= \int \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} dx = \int\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$$

$$= \frac{\pi}{4} \cdot \int dx + \frac{1}{2}\int x\, dx = \frac{\pi x}{4} + \frac{1}{4}x^2 + C.$$

EXAMPLE 12 *Evaluate* $\int\left(\frac{1+\sin x}{1-\sin x}\right) dx.$ **[CBSE 2008C]**

SOLUTION $I = \int \frac{(1+\sin x)}{(1-\sin x)} \times \frac{(1+\sin x)}{(1+\sin x)}\, dx$

$$= \int \frac{(1+\sin x)^2}{(1-\sin^2 x)}\,dx = \int \frac{(1+\sin^2 x+2\sin x)}{\cos^2 x}\,dx$$

$$= \int \left(\frac{1}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}+\frac{2\sin x}{\cos^2 x}\right)dx = \int (\sec^2 x+\tan^2 x+2\sec x\tan x)\,dx$$

$$= \int (2\sec^2 x - 1 + 2\sec x\tan x)\,dx$$

$$= 2\int \sec^2 x\,dx - \int dx + 2\int \sec x\tan x\,dx$$

$$= 2\tan x - x + 2\sec x + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{(\sin^6 x+\cos^6 x)}{\sin^2 x\cos^2 x}dx.$ **[CBSE 2014]**

SOLUTION Using the formula, $(a^3+b^3) = (a+b)^3 - 3ab(a+b)$, we get

$$I = \int \frac{(\sin^2 x+\cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{(1-3\sin^2 x\cos^2 x)}{\sin^2 x\cos^2 x}\,dx = \int \left\{\frac{1}{\sin^2 x\cos^2 x} - 3\right\}dx$$

$$= \int \left\{\frac{(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x} - 3\right\}dx = \int \left\{\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x} - 3\right\}dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 3)\,dx = \tan x - \cot x - 3x + C.$$

EXERCISE 12

Very-Short-Answer Questions

Evaluate:

1. (i) $\int x^7 dx$ (ii) $\int x^{-7} dx$ (iii) $\int x^{-1} dx$

   (iv) $\int x^{5/3} dx$ (v) $\int x^{-5/4} dx$ (vi) $\int 2^x\,dx$

   (vii) $\int \sqrt[3]{x^2}\,dx$ (viii) $\int \frac{1}{\sqrt[4]{x^3}}dx$ (ix) $\int \frac{2}{x^2}\,dx$

2. (i) $\int \left(6x^5 - \frac{2}{x^4} - 7x + \frac{3}{x} - 5 + 4e^x + 7^x\right)dx$

   (ii) $\int \left(8 - x + 2x^3 - \frac{6}{x^3} + 2x^{-5} + 5x^{-1}\right)dx$ (iii) $\int \left(\frac{x}{a}+\frac{a}{x}+x^a+a^x+ax\right)dx$

3. (i) $\int (2-5x)(3+2x)(1-x)\,dx$ (ii) $\int \sqrt{x}(ax^2+bx+c)\,dx$

   (iii) $\int \left(\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1\right)dx$

4. (i) $\int\left(x^2-\frac{1}{x^2}\right)^3 dx$ (ii) $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx$

(iii) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$ (iv) $\int\frac{(1+2x)^3}{x^4}\,dx$

(v) $\int\frac{(1+x)^3}{\sqrt{x}}\,dx$ (vi) $\int\frac{2x^2+x-2}{(x-2)}\,dx$

5. $\int\left[1+\frac{1}{(1+x^2)}-\frac{2}{\sqrt{1-x^2}}+\frac{5}{x\sqrt{x^2-1}}+a^x\right]dx$

6. (i) $\int\left(\frac{x^2-1}{x^2+1}\right)dx$ (ii) $\int\left(\frac{x^6-1}{x^2+1}\right)dx$

(iii) $\int\left(\frac{x^4}{1+x^2}\right)dx$ (iv) $\int\left(\frac{x^2}{1+x^2}\right)dx$

7. $\int\left(9\sin x-7\cos x-\frac{6}{\cos^2 x}+\frac{2}{\sin^2 x}+\cot^2 x\right)dx$

8. $\int\left(\frac{\cot x}{\sin x}-\tan^2 x-\frac{\tan x}{\cos x}+\frac{2}{\cos^2 x}\right)dx$

9. (i) $\int\sec x(\sec x+\tan x)\,dx$ **[CBSE 2011]**

(ii) $\int\text{cosec}\,x(\text{cosec}\,x-\cot x)\,dx$

10. (i) $\int(\tan x+\cot x)^2 dx$ (ii) $\int\left(\frac{1+2\sin x}{\cos^2 x}\right)dx$ (iii) $\int\left(\frac{3\cos x+4}{\sin^2 x}\right)dx$

11. (i) $\int\frac{1}{(1-\cos x)}\,dx$ (ii) $\int\frac{1}{(1-\sin x)}\,dx$ **[CBSE 2002C]**

12. (i) $\int\frac{\tan x}{(\sec x+\tan x)}\,dx$ (ii) $\int\frac{\text{cosec}\,x}{(\text{cosec}\,x-\cot x)}\,dx$

13. (i) $\int\frac{\cos x}{1+\cos x}\,dx$ (ii) $\int\frac{\sin x}{(1-\sin x)}\,dx$

14. (i) $\int\sqrt{1+\cos 2x}\,dx$ (ii) $\int\sqrt{1-\cos 2x}\,dx$

15. (i) $\int\frac{1}{(1+\cos 2x)}\,dx$ (ii) $\int\frac{1}{(1-\cos 2x)}\,dx$

16. $\int\sqrt{1+\sin 2x}\,dx$ 17. $\int\left(\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}\right)dx$

18. $\int\tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$ 19. $\int\cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)dx$

20. $\int \cos^{-1}(\sin x)\,dx$

21. $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$ **[CBSE 2003, '06C]**

22. $\int (3\cot x - 2\tan x)^2\,dx$

23. $\int (3\sin x + 4\operatorname{cosec} x)^2\,dx$

24. $\int \frac{dx}{(\sqrt{x+1}+\sqrt{x+2})}$ **[CBSE 2002]**

25. $\int \frac{dx}{(\sqrt{x+3}-\sqrt{x+2})}$ **[CBSE 2002]**

26. $\int \left(\frac{1+\cos x}{1-\cos x}\right)dx$

27. $\int \frac{(1+\tan x)}{(1-\tan x)}\,dx$

28. $\int \frac{\cos(x+a)}{\sin(x+b)}\,dx$ **[CBSE 2006C]**

29. $\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)}\,dx$ **[CBSE 2013]**

30. $\int (1-x)\sqrt{x}\,dx$ **[CBSE 2012]**

31. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\,dx$ **[CBSE 2012C]**

32. $\int \left\{\frac{2-3\sin x}{\cos^2 x}\right\}dx$ **[CBSE 2011]**

ANSWERS (EXERCISE 12)

1. (i) $\frac{x^8}{8}+C$ (ii) $-\frac{1}{6x^6}+C$ (iii) $\log|x|+C$

(iv) $\frac{3}{8}x^{8/3}+C$ (v) $-4x^{-1/4}+C$ (vi) $\frac{2^x}{\log 2}+C$

(vii) $\frac{3}{5}x^{5/3}+C$ (viii) $4x^{1/4}+C$ (ix) $\frac{-2}{x}+C$

2. (i) $x^6+\frac{2}{3x^3}-\frac{7x^2}{2}+3\log|x|-5x+4e^x+\frac{7^x}{\log 7}+C$

(ii) $8x-\frac{x^2}{2}+\frac{x^4}{2}+\frac{3}{x^2}-\frac{1}{2x^4}+5\log|x|+C$

(iii) $\frac{x^2}{2a}+a\log|x|+\frac{x^{(a+1)}}{(a+1)}+\frac{a^x}{\log a}+\frac{ax^2}{2}+C$

3. (i) $6x-\frac{17x^2}{2}+\frac{x^3}{3}+\frac{5x^4}{2}+C$ (ii) $\frac{2ax^{7/2}}{7}+\frac{2bx^{5/2}}{5}+\frac{2cx^{3/2}}{3}+C$

(iii) $\frac{2}{3}x^{3/2}-\frac{3}{7}x^{7/3}+21x^{1/3}-6e^x+x+C$

4. (i) $\frac{x^7}{7}+\frac{1}{5x^5}-x^3-\frac{3}{x}+C$ (ii) $\frac{2}{3}x^{3/2}-2x^{1/2}+C$

(iii) $\frac{x^2}{2}+\log|x|+2x+C$ (iv) $-\frac{1}{3x^3}+8\log|x|-\frac{3}{x^2}-\frac{12}{x}+C$

(v) $2\sqrt{x}+\frac{2}{7}x^{7/2}+2x^{3/2}+\frac{6}{5}x^{5/2}+C$ (vi) $x^2+5x+8\log|x-2|+C$

5. $x+\tan^{-1}x-2\sin^{-1}x+5\sec^{-1}x+\frac{a^x}{\log a}+C$

6. (i) $x - 2\tan^{-1}x + C$ (ii) $\frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1}x + C$

(iii) $\frac{x^3}{3} - x + \tan^{-1}x + C$ (iv) $x - \tan^{-1}x + C$

7. $-9\cos x - 7\sin x - 6\tan x - 3\cot x - x + C$

8. $-\text{cosec}\, x + \tan x + x - \sec x + C$

9. (i) $\tan x + \sec x + C$ (ii) $-\cot x + \text{cosec}\, x + C$

10. (i) $\tan x - \cot x + C$ (ii) $\tan x + 2\sec x + C$

(iii) $-3\,\text{cosec}\, x - 4\cot x + C$

11. (i) $-\cot x - \text{cosec}\, x + C$ (ii) $\tan x + \sec x + C$

12. (i) $\sec x - \tan x + x + C$ (ii) $-\cot x - \text{cosec}\, x + C$

13. (i) $-\text{cosec}\, x + \cot x + x + C$ (ii) $\sec x + \tan x - x + C$

14. (i) $\sqrt{2}\sin x + C$ (ii) $-\sqrt{2}\cos x + C$

15. (i) $\frac{1}{2}\tan x + C$ (ii) $-\frac{1}{2}\cot x + C$

16. $\sin x - \cos x + C$ **17.** $\sec x - \text{cosec}\, x + C$

18. $\frac{x^2}{2} + C$ **19.** $x^2 + C$

20. $\left(\frac{\pi x}{2} - \frac{x^2}{2} + C\right)$ **21.** $\frac{\pi x}{4} - \frac{x^2}{4} + C$

22. $4\tan x - 9\cos x - 25x + C$ **23.** $\frac{57}{2}x - \frac{9}{4}\sin 2x - 16\cot x + C$

24. $\frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} + C$ **25.** $\frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x+2)^{3/2} + C$

26. $-2\cot\frac{x}{2} - x + C$ **27.** $-\log|\cos x - \sin x| + C$

28. $\cos(a-b)\log|\sin(x+b)| - x\sin(a-b) + C$

29. $x\cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C$ **30.** $\frac{2}{15}x\sqrt{x}\,(5-3x) + C$

31. $\tan x - x + C$ **32.** $2\tan x - 3\sec x + C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 12)

21. $I = \int \tan^{-1}\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}}\,dx$

$= \int \tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\,dx = \int \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}dx$

$$= \int \left(\frac{\pi}{4} - \frac{x}{2} \right) dx = \frac{\pi x}{4} - \frac{x^2}{4} + C.$$

22. $I = \int (9 \cot^2 x + 4 \tan^2 x - 12)\, dx$

$$= \int \{9(\operatorname{cosec}^2 x - 1) + 4(\sec^2 x - 1) - 12\} dx.$$

23. $I = \int (9 \sin^2 x + 16 \operatorname{cosec}^2 x + 24)\, dx$

$$= \int \left\{ 9\left(\frac{1 - \cos 2x}{2} \right) + 16 \operatorname{cosec}^2 x + 24 \right\} dx$$

$$= \int \left(\frac{57}{2} - \frac{9}{2} \cos 2x + 16 \operatorname{cosec}^2 x \right) dx.$$

24. $I = \int \frac{1}{(\sqrt{x+2} + \sqrt{x+1})} \times \frac{(\sqrt{x+2} - \sqrt{x+1})}{(\sqrt{x+2} - \sqrt{x+1})}\, dx$

$$= \int \sqrt{x+2}\, dx - \int \sqrt{x+1}\, dx = \frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} + C.$$

26. $I = \int \frac{(1 + \cos x)}{(1 - \cos x)}\, dx = \int \frac{2\cos^2(x/2)}{2\sin^2(x/2)}\, dx$

$$= \int \cot^2(x/2)\, dx = \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1 \right) dx$$

$$= \frac{-\cot(x/2)}{(1/2)} - x + C = -2\cot\frac{x}{2} - x + C.$$

27. $I = \int \frac{\left(1 + \frac{\sin x}{\cos x}\right)}{\left(1 - \frac{\sin x}{\cos x}\right)} dx = \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx$

$$= -\int \frac{dt}{t}, \text{ where } (\cos x - \sin x) = t$$

$$= -\log|t| + C = -\log|\cos x - \sin x| + C.$$

28. $I = \int \frac{\cos(x + b + a - b)}{\sin(x + b)}\, dx$

$$= \int \frac{\cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)}{\sin(x+b)}\, dx$$

$$= \cos(a-b) \int \cot(x+b)\, dx - \sin(a-b) \int dx.$$

29. $I = \int \frac{\sin(x + \alpha - 2\alpha)}{\sin(x + \alpha)}\, dx$

$$= \int \frac{\sin(x+\alpha)\cos 2\alpha - \cos(x+\alpha)\sin 2\alpha}{\sin(x+\alpha)}\, dx = \cos 2\alpha \int dx - \sin 2\alpha \cdot \int \frac{\cos(x+\alpha)}{\sin(x+\alpha)}\, dx$$

$$= x\cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C.$$

32. $I = \int \left\{ \frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right\} dx = \int (2\sec^2 x - 3\sec x \tan x)\, dx.$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $\int x^6 dx = ?$

   (a) $7x^7 + C$ (b) $\frac{x^7}{7} + C$ (c) $6x^5 + C$ (d) $6x^7 + C$

2. $\int x^{5/3} dx = ?$

   (a) $\frac{3}{5}x^{2/3} + C$ (b) $\frac{8}{3}x^{8/3} + C$ (c) $\frac{3}{8}x^{8/3} + C$ (d) $\frac{5}{3}x^{8/3} + C$

3. $\int \frac{1}{x^3} dx = ?$

   (a) $\frac{-3}{x^2} + C$ (b) $\frac{-1}{2x^2} + C$ (c) $\frac{-1}{3x^2} + C$ (d) $\frac{x^{-2}}{2} + C$

4. $\int \sqrt[3]{x}\, dx = ?$

   (a) $\frac{3}{4}x^{3/4} + C$ (b) $\frac{4}{3}x^{3/4} + C$ (c) $\frac{3}{4}x^{4/3} + C$ (d) $\frac{4}{3}x^{4/3} + C$

5. $\int \frac{1}{\sqrt[3]{x}} dx = ?$

   (a) $\frac{3}{2}x^{2/3} + C$ (b) $\frac{3}{2x^{2/3}} + C$ (c) $\frac{2}{3x^{2/3}} + C$ (d) $\frac{2}{3}x^{3/2} + C$

6. $\int \sqrt[3]{x^2}\, dx = ?$

   (a) $\frac{5}{3}x^{5/3} + C$ (b) $\frac{3}{5}x^{5/3} + C$ (c) $\frac{5}{3}x^{3/5} + C$ (d) $\frac{3}{5}x^{3/5} + C$

7. $\int 3^x dx = ?$

   (a) $3^x(\log\ 3) + C$ (b) $3^x + C$ (c) $\frac{3^x}{\log\ 3} + C$ (d) $\frac{\log\ 3}{3^x} + C$

8. $\int 2^{\log x} dx = ?$

   (a) $\frac{2^{\log x + 1}}{(\log\ x + 1)} + C$ (b) $\frac{x^{(\log 2 + 1)}}{(\log\ 2 + 1)} + C$

   (c) $\frac{2^{\log x}}{\log\ 2} + C$ (d) $\frac{2^{\log x}}{2} + C$

9. $\int \text{cosec}\, x(\text{cosec}\, x + \cot\, x)\, dx = ?$

   (a) $\cot\, x - \text{cosec}\, x + C$ (b) $-\cot\, x + \text{cosec}\, x + C$

   (c) $\cot\, x + \text{cosec}\, x + C$ (d) $-\cot\, x - \text{cosec}\, x + C$

10. $\int \frac{\sec x}{(\sec x + \tan x)} dx = ?$

(a) $\tan x + \sec x + C$ (b) $\tan x - \sec x + C$
(c) $-\tan x + \sec x + C$ (d) $-\tan x - \sec x + C$

11. $\int \frac{(1-\cos 2x)}{(1+\cos 2x)} dx = ?$

(a) $\tan x + x + C$ (b) $\tan x - x + C$
(c) $-\tan x + x + C$ (d) $-\tan x - x + C$

12. $\int \frac{1}{\sin^2 x \cos^2 x} dx = ?$

(a) $\tan x + \cot x + C$ (b) $-\tan x + \cot x + C$
(c) $\tan x - \cot x + C$ (d) none of these

13. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx = ?$

(a) $-\cot x - \tan x + C$ (b) $-\cot x + \tan x + C$
(c) $\cot x - \tan x + C$ (d) $\cot x + \tan x + C$

14. $\int \frac{(\cos 2x - \cos 2\alpha)}{(\cos x - \cos \alpha)} dx = ?$

(a) $2\sin x + 2x\cos \alpha + C$ (b) $2\sin x - 2x\cos \alpha + C$
(c) $-2\sin x + 2x\cos \alpha + C$ (d) $-2\sin x - 2x\cos \alpha + C$

15. $\int \sqrt{1+\cos 2x}\, dx = ?$

(a) $\sqrt{2}\cos x + C$ (b) $\sqrt{2}\sin x + C$ (c) $-\sqrt{2}\cos x + C$ (d) $-\sqrt{2}\sin x + C$

16. $\int \sqrt{1+\sin 2x}\, dx = ?$

(a) $\sin x + \cos x + C$ (b) $-\sin x + \cos x + C$
(c) $\sin x - \cos x + C$ (d) $-\sin x - \cos x + C$

17. $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx = ?$

(a) $\cot x + \tan x + C$ (b) $-\cot x + \tan x + C$
(c) $\cot x - \tan x + C$ (d) $-\cot x - \tan x + C$

18. $\int \frac{dx}{(1-\cos 2x)} = ?$

(a) $\frac{1}{2}\cot x + C$ (b) $2\cot x + C$
(c) $-\frac{1}{2}\cot x + C$ (d) $-2\cot x + C$

19. $\int \frac{\sin 2x}{\sin x} dx = ?$

(a) $2\sin x + C$ (b) $\frac{1}{2}\sin x + C$ (c) $2\cos x + C$ (d) $\frac{1}{2}\cos x + C$

20. $\int \frac{(1-\sin x)}{\cos^2 x} dx = ?$

(a) $\tan x + \sec x + C$ (b) $\tan x - \sec x + C$
(c) $-\tan x + \sec x + C$ (d) $-\tan x - \sec x + C$

21. $\int \cot^2 x \, dx = ?$

(a) $-\cot x - x + C$ (b) $\cot x - x + C$
(c) $-\cot x + x + C$ (d) $\cot x + x + C$

22. $\int \sec x(\sec x + \tan x) dx = ?$

(a) $\tan x - \sec x + C$ (b) $-\tan x + \sec x + C$
(c) $\tan x + \sec x + C$ (d) $-\tan x - \sec x + C$

23. $\int \frac{\sec^2 x}{\text{cosec}^2 x} dx = ?$

(a) $\tan x + x + C$ (b) $\tan x - x + C$
(c) $-\tan x + x + C$ (d) $-\tan x + x + C$

24. $\int \frac{\sin^2 x}{(1+\cos x)} dx = ?$

(a) $x + \sin x + C$ (b) $x - \sin x + C$
(c) $\sin x - x + C$ (d) $-\sin x - x + C$

25. $\int \frac{\cot x}{(\text{cosec } x - \cot x)} dx = ?$

(a) $-\text{cosec } x - \cot x - x + C$ (b) $\text{cosec } x - \cot x - x + C$
(c) $-\text{cosec } x + \cot x - x + C$ (d) $\text{cosec } x + \cot x - x + C$

26. $\int \frac{\sin x}{(1+\sin x)} dx = ?$

(a) $\sec x + \tan x + x + C$ (b) $\sec x - \tan x + x + C$
(c) $-\sec x + \tan x + x + C$ (d) none of these

27. $\int \frac{(1+\sin x)}{(1-\sin x)} dx = ?$

(a) $2\tan x + 2\sec x + x + C$ (b) $2\tan x + 2\sec x - x + C$
(c) $\tan x + \sec x - x + C$ (d) none of these

28. $\int \frac{1}{(1+\cos x)} dx = ?$

(a) $-\cot x + \text{cosec } x + C$ (b) $\cot x - \text{cosec } x + C$
(c) $\cot x + \text{cosec } x + C$ (d) none of these

29. $\int \sin^{-1}(\cos x) dx = ?$

(a) $\text{cosec } x + C$ (b) $\frac{\pi x}{2} + \frac{x^2}{2} + C$ (c) $\frac{\pi x}{2} - \frac{x^2}{2} + C$ (d) $\frac{x^2}{2} - \frac{\pi x}{2} + C$

30. $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx = ?$

(a) $\frac{1}{(1+x^2)}+C$ (b) $\frac{1}{\sqrt{1+x^2}}+C$ (c) $\frac{1}{\sqrt{1-x^2}}+C$ (d) $\frac{x^2}{2}+C$

31. $\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right) dx = ?$

(a) $\frac{-1}{(1+x^2)}+C$ (b) $\frac{-1}{(1-x^2)}+C$ (c) $\frac{x^2}{2}+C$ (d) $2x^2+C$

32. $\int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right) dx = ?$

(a) $-x^2+C$ (b) x^2+C (c) $\frac{x^2}{2}+C$ (d) $2x^2+C$

33. $\int \cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) dx = ?$

(a) x^2+C (b) $-x^2+C$ (c) $\frac{1}{\sqrt{1+x^2}}+C$ (d) $\frac{1}{\sqrt{1-x^2}}+C$

34. $\int \tan^{-1}(\operatorname{cosec} x - \cot x) dx = ?$

(a) $\frac{x^2}{4}+C$ (b) $\frac{-x^2}{4}+C$ (c) $\frac{x^2}{2}+C$ (d) $\frac{-x^2}{2}+C$

35. $\int \left(\frac{(x^4+1)}{(x^2+1)}\right) dx = ?$

(a) $\frac{x^3}{3}+x-\tan^{-1}x+C$ (b) $\frac{x^3}{3}-x-2\tan^{-1}x+C$

(c) $\frac{x^3}{3}+x-2\tan^{-1}x+C$ (d) none of these

36. $\int \frac{(ax+b)}{(cx+d)} dx = ?$

(a) $\frac{ax}{c}+\log|cx+d|+C$ (b) $\frac{a}{c}+\log|cx+d|+C$

(c) $\frac{ax}{c}+\frac{(bc-ad)}{c^2}\log|cx+d|+C$ (d) none of these

37. $\int \frac{(\sin^3 x+\cos^3 x)}{\sin^2 x \cos^2 x} dx = ?$

(a) $\sin x - \cos x + C$ (b) $\tan x - \cos x + C$

(c) $\sec x - \operatorname{cosec} x + C$ (d) none of these

38. $\int \frac{\sin x}{\sin(x-\alpha)}dx = ?$

(a) $x\cos\alpha + (\sin\alpha)\log|\sin(x-\alpha)| + C$
(b) $x\sin\alpha + (\sin\alpha)\log|\sin(x-\alpha)| + C$
(c) $x\cos\alpha - (\sin\alpha)\log|\sin(x-\alpha)| + C$
(d) $x\sin\alpha - (\sin\alpha)\log|\sin(x-\alpha)| + C$

39. $\int \sin 3x \sin 2x\,dx = ?$

(a) $-\frac{1}{5}\cos 5x + C$ (b) $\frac{1}{2}\sin x + \frac{1}{10}\sin 5x - C$
(c) $\frac{1}{2}\sin x - \frac{1}{10}\sin 5x - C$ (d) $-\frac{1}{3}\cos 3x - \frac{1}{2}\sin 2x + C$

40. $\int \cos 3x \sin 2x\,dx = ?$

(a) $\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$ (b) $-\frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$
(c) $-\frac{1}{2}\cos x + \frac{1}{10}\cos 5x + C$ (d) none of these

41. $\int \cos 4x \cos x\,dx = ?$

(a) $\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x + C$ (b) $\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x + C$
(c) $\frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + C$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) **2.** (c) **3.** (b) **4.** (c) **5.** (a) **6.** (b) **7.** (c) **8.** (b) **9.** (d) **10.** (b)
11. (b) **12.** (c) **13.** (a) **14.** (a) **15.** (b) **16.** (c) **17.** (d) **18.** (c) **19.** (a) **20.** (b)
21. (b) **22.** (c) **23.** (b) **24.** (b) **25.** (a) **26.** (b) **27.** (b) **28.** (a) **29.** (c) **30.** (d)
31. (c) **32.** (b) **33.** (a) **34.** (a) **35.** (b) **36.** (c) **37.** (c) **38.** (a) **39.** (c) **40.** (a)
41. (c)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

8. $2^{\log x} = x^{\log 2}$.

10. $I = \int \left\{ \frac{\sec x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)} \right\} dx.$

12. Write $1 = (\sin^2 x + \cos^2 x)$.

13. $\cos 2x = (\cos^2 x - \sin^2 x)$.

14. $\cos 2x = (2\cos^2 x - 1)$ and $\cos 2\alpha = (2\cos^2\alpha - 1)$.

15. $\sqrt{1 + \cos 2x} = \sqrt{2\cos^2 x} = \sqrt{2}\cos x$.

16. $(1 + \sin 2x) = \sin^2 x + \cos^2 x + 2\sin x\cos x = (\sin x + \cos x)^2$

$\therefore \quad \sqrt{1 + \sin 2x} = (\sin x + \cos x)$.

17. $I = \int \frac{(\cos^2 x - \sin^2 x)}{(\sin^2 x \cos^2 x)} dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx$

$= \int (\text{cosec}^2 x - \sec^2 x)\, dx.$

18. $\frac{1}{(1 - \cos 2x)} = \frac{1}{2 \sin^2 x} = \frac{1}{2} \text{cosec}^2 x.$ **21.** $\cot^2 x = (\text{cosec}^2 x - 1).$

23. $\frac{\sec^2 x}{\text{cosec}^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x = (\sec^2 x - 1).$

25. $I = \int \left\{ \frac{\cot x}{(\text{cosec } x - \cot x)} \times \frac{(\text{cosec } x + \cot x)}{(\text{cosec } x + \cot x)} \right\} dx$

$= \int (\text{cosec } x \cot x + \cot^2 x) dx = \int (\text{cosec } x \cot x + \text{cosec}^2 x - 1) dx.$

26. $I = \int \left\{ \frac{\sin x}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)} \right\} dx = \int \frac{\sin x (1 - \sin x)}{(1 - \sin^2 x)} dx$

$= \int \frac{(\sin x - \sin^2 x)}{\cos^2 x} dx = \int \left(\frac{\sin x}{\cos^2 x} - \tan^2 x \right) dx$

$= \int (\sec x \tan x - \sec^2 x + 1)\, dx.$

27. $I = \int \left\{ \frac{(1 + \sin x)}{(1 - \sin x)} \times \frac{(1 - \sin x)}{(1 + \sin x)} \right\} dx = \int \frac{(1 + \sin x)^2}{(1 - \sin^2 x)} dx$

$= \int \frac{(1 + \sin^2 x + 2 \sin x)}{\cos^2 x} dx = \int (\sec^2 x + \tan^2 x + 2 \sec x \tan x)\, dx$

$= \int (2 \sec^2 x - 1 + 2 \sec x \tan x)\, dx.$

28. $I = \int \left\{ \frac{1}{(1 + \cos x)} \times \frac{(1 - \cos x)}{(1 - \cos x)} \right\} dx = \int \frac{(1 - \cos x)}{\sin^2 x} dx$

$= \left\{ \frac{-1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right\} dx = \int (\text{cosec}^2 x - \text{cosec } x \cot x)\, dx.$

29. $\sin^{-1}(\cos x) = \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - x \right) \right\} = \left(\frac{\pi}{2} - x \right).$

32. $\sin^{-1} \left(\frac{2 \tan x}{1 + \tan^2 x} \right) = \sin^{-1} (\sin 2x) = 2x.$

33. $\cos^{-1} \left(\frac{1 - \tan^2 x}{1 + \tan^2 x} \right) = \cos^{-1} \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \right)$

$= \cos^{-1} (\cos^2 x - \sin^2 x) = \cos^{-1} (\cos 2x) = 2x.$

34. $\tan^{-1} (\text{cosec } x - \cot x) = \tan^{-1} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \tan^{-1} \left(\frac{1 - \cos x}{\sin x} \right)$

$= \tan^{-1} \left\{ \frac{2 \sin^2 \left(\frac{x}{2} \right)}{2 \sin \left(\frac{x}{2} \right) \cos \left(\frac{x}{2} \right)} \right\} = \tan^{-1} \tan \left(\frac{x}{2} \right) = \frac{x}{2}.$

35. On dividing $(x^4 + 1)$ by $(x^2 + 1)$, we get

$$\frac{(x^4 + 1)}{(x^2 + 1)} = (x^2 - 1) + \frac{2}{(1 + x^2)}.$$

36. On dividing $(ax + b)$ by $(cx + d)$, we get

$$\int \frac{(ax + b)}{(cx + d)} dx = \int \left\{ \frac{a}{c} + \frac{(bc - ad)}{c(cx + d)} \right\} dx$$

$$= \int \frac{a}{c} dx + \frac{(bc - ad)}{c^2} \cdot \int \frac{c}{(cx + d)} dx$$

$$= \frac{ax}{c} + \frac{(bc - ad)}{c^2} \log |cx + d| + C.$$

37. $I = \int \frac{\sin^3 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^3 x}{\sin^2 x \cos^2 x} dx$

$= \int \sec x \tan x \, dx + \int \text{cosec}\, x \cot x \, dx$

$= \sec x - \text{cosec}\, x + C.$

38. $I = \frac{\sin(x - \alpha + \alpha)}{\sin(x - \alpha)} dx = \int \frac{\sin(x - \alpha)\cos\alpha + \cos(x - \alpha)\sin\alpha}{\sin(x - \alpha)} dx$

$= (\cos\alpha) \int dx + (\sin\alpha) \int \cot(x - \alpha) dx$

$= x\cos\alpha + (\sin\alpha) \log |\sin(x - \alpha)| + C.$

39. $I = \frac{1}{2} \int 2 \sin 3x \sin 2x \, dx$

$= \frac{1}{2} \int (\cos x - \cos 5x) dx = \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C.$

40. $I = \frac{1}{2} 2 \cos 3x \sin 2x \, dx$

$= \frac{1}{2} \int (\sin 5x - \sin x) dx = \frac{-1}{10} \cos 5x + \frac{1}{2} \cos x + C.$

41. $I = \frac{1}{2} 2 \cos 4x \cos x \, dx = \frac{1}{2} \int (\cos 5x + \cos 3x) dx$

$= \frac{1}{10} \sin 5x + \frac{1}{6} \sin 3x + C.$

13. METHODS OF INTEGRATION

Integration by Substitution

If we have to evaluate an integral of the type $\int f\{\phi(x)\}\cdot\phi'(x)\,dx$ then we put $\phi(x)=t$ and $\phi'(x)dx=dt$. With this substitution, the integrand becomes easily integrable.

Case I When the integrand is of the form $f(ax+b)$, we put $(ax+b)=t$ and $dx=\dfrac{1}{a}dt$.

Case II When the integrand is of the form $x^{n-1}\cdot f(x^n)$, we put $x^n=t$ and $nx^{n-1}dx=dt$.

Case III When the integrand is of the form $\{f(x)\}^n\cdot f'(x)$, we put $f(x)=t$ and $f'(x)\,dx=dt$.

Case IV When the integrand is of the form $\dfrac{f'(x)}{f(x)}$, we put $f(x)=t$ and $f'(x)\,dx=dt$.

THEOREM 1 $\int(ax+b)^n dx=\dfrac{(ax+b)^{n+1}}{a(n+1)}+C,\quad$ *where* $n\neq -1$.

PROOF Putting $ax+b=t$, we get $\quad a\,dx=dt$ or $dx=\dfrac{1}{a}dt$.

$$\therefore\quad \int(ax+b)^n dx=\frac{1}{a}\int t^n dt=\frac{1}{a}\cdot\frac{t^{n+1}}{(n+1)}+C=\frac{(ax+b)^{n+1}}{a(n+1)}+C.$$

THEOREM 2 *(i)* $\int\cos(ax+b)\,dx=\dfrac{1}{a}\sin(ax+b)+C$

(ii) $\int\text{cosec}^2(ax+b)\,dx=-\dfrac{1}{a}\cot(ax+b)+C$

PROOF (i) Put $(ax+b)=t$ so that $dx=\dfrac{1}{a}\,dt$.

$$\therefore\quad \int\cos(ax+b)\,dx=\frac{1}{a}\int\cos t\,dt$$

$$=\frac{1}{a}\sin t+C$$

$$=\frac{1}{a}\sin(ax+b)+C.$$

(ii) Put $(ax + b) = t$ so that $dx = \frac{1}{a}\, dt$.

$$\therefore \quad \int \text{cosec}^2(ax + b)\, dx = \frac{1}{a}\int \text{cosec}^2 t\, dt$$

$$= -\frac{1}{a}\cot t + C = -\frac{1}{a}\cot (ax + b) + C.$$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:* *(i)* $\int (3x+5)^7\, dx$ *(ii)* $\int (4-9x)^5\, dx$

(iii) $\int \frac{1}{(2-3x)^4}\, dx$ *(iv)* $\int \sqrt{ax+b}\, dx$

SOLUTION (i) Put $(3x+5) = t$ so that $3dx = dt$ or $dx = \frac{1}{3}\, dt$.

$$\therefore \quad \int (3x+5)^7\, dx = \frac{1}{3}\int t^7 dt = \frac{1}{3}\cdot\frac{t^8}{8} + C = \frac{(3x+5)^8}{24} + C.$$

(ii) Put $(4 - 9x) = t$ so that $-9\, dx = dt$ or $dx = -\frac{1}{9}\, dt$.

$$\therefore \quad \int (4-9x)^5 dx = -\frac{1}{9}\int t^5 dt = -\frac{1}{9}\cdot\frac{t^6}{6} + C = \frac{-(4-9x)^6}{54} + C.$$

(iii) Put $(2 - 3x) = t$ so that $-3\, dx = dt$ or $dx = -\frac{1}{3}\, dt$.

$$\therefore \quad \int \frac{1}{(2-3x)^4} dx = -\frac{1}{3}\int \frac{1}{t^4} dt = -\frac{1}{3}\cdot\frac{1}{(-3t^3)} + C = \frac{1}{9(2-3x)^3} + C.$$

(iv) Put $(ax + b) = t$ so that $a\, dx = dt$.

$$\therefore \quad \int \sqrt{ax+b}\, dt = \frac{1}{a}\int \sqrt{t}\, dt = \frac{2}{3a}t^{3/2} + C = \frac{2(ax+b)^{3/2}}{3a} + C.$$

EXAMPLE 2 *Evaluate:* *(i)* $\int \cos 2x\, dx$ *(ii)* $\int e^{(5x+3)} dx$

(iii) $\int \sec^2(3x+5)\, dx$ *(iv)* $\int \sin^3 x\, dx$

SOLUTION (i) Put $2x = t$ so that $2dx = dt$ or $dx = \frac{1}{2} dt$.

$$\therefore \quad \int \cos 2x\, dx = \frac{1}{2}\int \cos t\, dt = \frac{1}{2}\sin t + C = \frac{1}{2}\sin 2x + C.$$

(ii) Put $(5x + 3) = t$ so that $5dx = dt$ or $dx = \frac{1}{5} dt$.

$$\therefore \quad \int e^{(5x+3)} dx = \frac{1}{5}\int e^t dt = \frac{1}{5}\cdot e^t + C = \frac{1}{5}e^{(5x+3)} + C.$$

(iii) Put $(3x+5)=t$ so that $3dx=dt$ or $dx=\frac{1}{3}dt$.

$$\therefore \int \sec^2(3x+5)\,dx=\frac{1}{3}\int \sec^2 t\,dt=\frac{1}{3}\tan t+C$$
$$=\frac{1}{3}\tan(3x+5)+C.$$

(iv) We know that $\sin 3x=3\sin x-4\sin^3 x$.

$$\therefore \quad \sin^3 x=\frac{1}{4}(3\sin x-\sin 3x).$$

$$\text{So, } \int \sin^3 x\,dx=\int\left(\frac{3}{4}\sin x-\frac{1}{4}\sin 3x\right)dx$$
$$=\frac{3}{4}\int \sin x\,dx-\frac{1}{4}\int \sin 3x\,dx$$
$$=\frac{3}{4}(-\cos x)-\frac{1}{4}\cdot\frac{(-\cos 3x)}{3}+C$$
$$=-\frac{3}{4}\cos x+\frac{\cos 3x}{12}+C.$$

EXAMPLE 3 *Evaluate:* *(i)* $\int \frac{\log x}{x}\,dx$ *(ii)* $\int \frac{\sec^2(\log x)}{x}\,dx$

(iii) $\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\,dx$ *(iv)* $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

SOLUTION (i) Put $\log x=t$ so that $\frac{1}{x}dx=dt$.

$$\therefore \int \frac{\log x}{x}\,dx=\int t\,dt=\frac{1}{2}t^2+C=\frac{1}{2}(\log x)^2+C.$$

(ii) Put $\log x=t$ so that $\frac{1}{x}dx=dt$.

$$\therefore \int \frac{\sec^2(\log x)}{x}\,dx=\int \sec^2 t\,dt=\tan t+C=\tan(\log x)+C.$$

(iii) Put $\tan^{-1}x=t$ so that $\frac{1}{(1+x^2)}dx=dt$.

$$\therefore \int \frac{e^{\tan^{-1}x}}{(1+x^2)}\,dx=\int e^t\,dt=e^t+C=e^{\tan^{-1}x}+C.$$

(iv) Put $\sqrt{x}=t$ so that $\frac{1}{2}x^{-1/2}dx=dt$

or $\frac{1}{\sqrt{x}}dx=2\,dt$.

$$\therefore \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sin t\,dt$$

$$= 2(-\cos t) + C = -2\cos t + C = -2\cos\sqrt{x} + C.$$

EXAMPLE 4 *Evaluate:* *(i)* $\int \cos^3 x \sin x\,dx$ *(ii)* $\int (\sqrt{\sin x})\cos x\,dx$

(iii) $\int \frac{\text{cosec}^2 x}{(1+\cot x)}\,dx$ *(iv)* $\int \frac{\sin x}{(3+4\cos x)^2}\,dx$

SOLUTION (i) Put $\cos x = t$ so that $\sin x\,dx = -dt$.

$$\therefore \int \cos^3 x \sin x\,dx = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{1}{4}\cos^4 x + C.$$

(ii) Put $\sin x = t$ so that $\cos x\,dx = dt$.

$$\therefore \int (\sqrt{\sin x})\cos x\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{3/2} + C = \frac{2}{3}(\sin x)^{3/2} + C.$$

(iii) Put $(1+\cot x) = t$ so that $-\text{cosec}^2 x\,dx = dt$.

$$\therefore \int \frac{\text{cosec}^2 x}{(1+\cot x)}\,dx = -\int \frac{1}{t}\,dt$$

$$= -\log t + C = -\log|(1+\cot x)| + C.$$

(iv) Put $(3 + 4\cos x) = t$ so that $-4\sin x\,dx = dt$.

$$\therefore \int \frac{\sin x}{(3+4\cos x)^2}dx = -\frac{1}{4}\int \frac{1}{t^2}dt = \frac{1}{4t} + C = \frac{1}{4(3+4\cos x)} + C.$$

EXAMPLE 5 *Evaluate:* *(i)* $\int \frac{2x}{(2x+1)^2}\,dx$ *(ii)* $\int \frac{(2+3x)}{(3-2x)}\,dx$

SOLUTION (i) Put $(2x + 1) = t$ so that $2x = (t - 1)$ and $dx = \frac{1}{2}\,dt$.

$$\therefore \int \frac{2x}{(2x+1)^2}\,dx = \frac{1}{2}\int \frac{(t-1)}{t^2}\,dt$$

$$= \frac{1}{2}\int \frac{1}{t}\,dt - \frac{1}{2}\int \frac{1}{t^2}\,dt = \frac{1}{2}\log|t| + \frac{1}{2t} + C$$

$$= \frac{1}{2}\log|(2x+1)| + \frac{1}{2(2x+1)} + C.$$

(ii) Put $(3-2x) = t$ so that $x = \left(\frac{3-t}{2}\right)$ and $dx = -\frac{1}{2}\,dt$.

$$\therefore \int \frac{(2+3x)}{(3-2x)}\,dt = -\frac{1}{2}\int \frac{\left[2+\left(\frac{9-3t}{2}\right)\right]}{t}\,dt = -\frac{1}{4}\int \frac{(13-3t)}{t}\,dt$$

$$= \frac{-13}{4}\int \frac{1}{t}\,dt + \frac{3}{4}\int dt = -\frac{13}{4}\log|t| + \frac{3}{4}t + C$$

$$= -\frac{13}{4}\log|(3-2x)| + \frac{3}{4}(3-2x) + C.$$

EXAMPLE 6 *Evaluate:*

$$(i)\ \int \frac{3x^2}{(1+x^6)}\,dx \qquad (ii)\ \int \frac{x^3}{(x^2+1)^3}\,dx \qquad (iii)\ \frac{x^8}{(1-x^3)^{1/3}}\,dx$$

SOLUTION (i) Put $x^3 = t$ so that $3x^2\,dx = dt$.

$$\therefore\ \int \frac{3x^2}{(1+x^6)}\,dx = \int \frac{dt}{(1+t^2)} = \tan^{-1}t + C = \tan^{-1}x^3 + C.$$

(ii) Put $(x^2+1) = t$ so that $x^2 = (t-1)$ and $x\,dx = \frac{1}{2}\,dt$.

$$\therefore\ \int \frac{x^3}{(x^2+1)^3}\,dx = \int \frac{x^2 \cdot x}{(x^2+1)^3}\,dx$$

$$= \frac{1}{2}\int \frac{(t-1)}{t^3}\,dt = \frac{1}{2}\int \frac{1}{t^2}\,dt - \frac{1}{2}\int \frac{1}{t^3}\,dt$$

$$= \frac{-1}{2t} + \frac{1}{4t^2} + C = \frac{-1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C$$

$$= \frac{-(1+2x^2)}{4(x^2+1)^2} + C.$$

(iii) Put $(1-x^3) = t$ so that $x^3 = (1-t)$ and $x^2dx = -\frac{1}{3}\,dt$.

$$\therefore\ \int \frac{x^8}{(1-x^3)^{1/3}}\,dx = \int \frac{x^6 \cdot x^2}{(1-x^3)^{1/3}}\,dx$$

$$= -\frac{1}{3}\int \frac{(1-t)^2}{t^{1/3}}\,dt = -\frac{1}{3}\int \frac{(1+t^2-2t)}{t^{1/3}}\,dt$$

$$= -\frac{1}{3}\int t^{-1/3}\,dt - \frac{1}{3}\int t^{5/3}dt + \frac{2}{3}\int t^{2/3}dt$$

$$= -\frac{1}{2}t^{2/3} - \frac{1}{8}t^{8/3} + \frac{2}{5}t^{5/3} + C$$

$$= -\frac{1}{2}(1-x^3)^{2/3} - \frac{1}{8}(1-x^3)^{8/3} + \frac{2}{5}(1-x^3)^{5/3} + C.$$

EXAMPLE 7 *Evaluate* $\displaystyle\int \frac{dx}{x\cdot\sqrt{x^6-1}}$.

SOLUTION Put $x^3 = t$ so that $3x^2dx = dt$ or $x^2dx = \frac{1}{3}\,dt$.

$$\therefore\ \int \frac{dx}{x\cdot\sqrt{x^6-1}} = \int \frac{x^2}{x^3\cdot\sqrt{x^6-1}}\,dx$$

[multiplying numerator and denominator by x^2]

$$= \frac{1}{3}\int \frac{1}{t\sqrt{t^2-1}}\,dt = \frac{1}{3}\sec^{-1}t + C = \frac{1}{3}\sec^{-1}x^3 + C.$$

EXAMPLE 8 *Evaluate* $\int \frac{1}{(\sqrt{x}+x)}\,dx.$ **[CBSE 2003]**

SOLUTION $\int \frac{1}{(\sqrt{x}+x)}\,dx = \int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$

Now, put $(1+\sqrt{x}) = t$ so that $\frac{1}{\sqrt{x}}\,dx = 2\,dt.$

$$\therefore \int \frac{1}{(\sqrt{x}+x)}\,dx = \int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$$

$$= 2\int \frac{1}{t}\,dt = 2\log|t| + C = 2\log|(1+\sqrt{x})| + C.$$

EXAMPLE 9 *Evaluate:* *(i)* $\int \frac{(x-1)}{\sqrt{x+4}}\,dx$ *(ii)* $\int x\sqrt{x+2}\,dx$

(iii) $\int (4x+2)\sqrt{x^2+x+1}\,dx$ *(iv)* $\int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}\,dx$

SOLUTION (i) Put $(x+4) = t^2$ so that $x = (t^2-4)$ and $dx = 2t\,dt.$

$$\therefore \int \frac{(x-1)}{\sqrt{x+4}}\,dx = 2\int \frac{(t^2-5)t}{t}\,dt$$

$$= 2\int t^2 dt - 10\int dt = \frac{2t^3}{3} - 10t + C$$

$$= \frac{2}{3}(x+4)^{3/2} - 10(x+4)^{1/2} + C.$$

(ii) Put $(x+2) = t^2$ so that $x = (t^2-2)$ and $dx = 2t\,dt.$

$$\therefore \int x\sqrt{x+2}\,dx = \int (t^2-2)2t^2 dt = 2\int t^4 dt - 4\int t^2 dt$$

$$= \frac{2t^5}{5} - \frac{4t^3}{3} + C = \frac{2(x+2)^{5/2}}{5} - \frac{4(x+2)^{3/2}}{3} + C.$$

(iii) Put $(x^2+x+1) = t$ so that $(2x+1)\,dx = dt.$

$$\therefore \int (4x+2)(\sqrt{x^2+x+1})\,dx = 2\int \sqrt{t}\,dt$$

$$= \frac{4}{3}t^{3/2} + C = \frac{4}{3}(x^2+x+1)^{3/2} + C.$$

(iv) Put $(2x^2+3x+1) = t$ so that $(4x+3)\,dx = dt.$

$$\therefore \int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}\,dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{2x^2+3x+1} + C.$$

EXAMPLE 10 *Evaluate:*

(i) $\int \frac{(2x+5)}{(x^2+5x+9)}\,dx$ *(ii)* $\int \frac{(6x-7)}{(3x^2-7x+5)^2}\,dx$

(iii) $\int \frac{(\cos x-\sin x)}{(\cos x+\sin x)}\,dx$ *(iv)* $\int \frac{\sec x}{\log(\sec x+\tan x)}\,dx$

SOLUTION (i) Put $(x^2+5x+9)=t$ so that $(2x+5)\,dx=dt$.

$$\therefore \int \frac{(2x+5)}{(x^2+5x+9)}\,dx=\int \frac{1}{t}\,dt=\log|t|+C$$

$$=\log|(x^2+5x+9)|+C.$$

(ii) Put $(3x^2-7x+5)=t$ so that $(6x-7)\,dx=dt$.

$$\therefore \int \frac{(6x-7)}{(3x^2-7x+5)^2}\,dx=\int \frac{1}{t^2}\,dt=-\frac{1}{t}+C=\frac{-1}{(3x^2-7x+5)}+C.$$

(iii) Put $(\cos x+\sin x)=t$ so that $(\cos x-\sin x)\,dx=dt$.

$$\therefore \int \frac{(\cos x-\sin x)}{(\cos x+\sin x)}\,dx=\int \frac{1}{t}\,dt$$

$$=\log|t|+C=\log|(\cos x+\sin x)|+C.$$

(iv) Put $\log(\sec x+\tan x)=t$.

Then, on differentiation, we get

$$\frac{1}{(\sec x+\tan x)}\cdot(\sec x\tan x+\sec^2 x)\,dx=dt$$

or $\sec x\,dx=dt$.

$$\therefore \int \frac{\sec x}{\log(\sec x+\tan x)}\,dx=\int \frac{1}{t}\,dt$$

$$=\log|t|+C=\log|\log(\sec x+\tan x)|+C.$$

EXAMPLE 11 *Evaluate* $\int \frac{\sin 2x}{(a^2\sin^2 x+b^2\cos^2 x)}\,dx.$ **[CBSE 2005]**

SOLUTION Put $(a^2\sin^2 x+b^2\cos^2 x)=t$ so that

$$2(a^2-b^2)\sin x\cos x\,dx=dt \;\Leftrightarrow\; \sin 2x\,dx=\frac{dt}{(a^2-b^2)}.$$

$$\therefore I=\int \frac{\sin 2x}{(a^2\sin^2 x+b^2\cos^2 x)}\,dx=\frac{dt}{(a^2-b^2)t}$$

$$=\frac{1}{(a^2-b^2)}\log|t|+C$$

$$=\frac{1}{(a^2-b^2)}\log(a^2\sin^2 x+b^2\cos^2 x)+C.$$

EXAMPLE 12 *Evaluate* $\int \frac{x^2\tan^{-1}x^3}{(1+x^6)}\,dx.$

SOLUTION Put $\tan^{-1}x^3 = t$ so that $\frac{3x^2}{(1+x^6)}\,dx = dt$ or $\frac{x^2}{(1+x^6)}\,dx = \frac{1}{3}dt$

$$\therefore \int \frac{x^2 \tan^{-1}x^3}{(1+x^6)}\,dx = \frac{1}{3}\int t\,dt = \frac{1}{6}t^2 + C = \frac{1}{6}(\tan^{-1}x^3)^2 + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx.$

SOLUTION
$$\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx = \int \frac{\tan x}{(\sqrt{\tan x})\cdot \sin x \cos x}\,dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$$
$$= \int \frac{1}{\sqrt{t}}\,dt, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$
$$= 2\sqrt{t} + C = 2\sqrt{\tan x} + C.$$

EXAMPLE 14 *Evaluate* $\int \frac{dx}{(\sqrt{2x+3} + \sqrt{2x-3})}.$

SOLUTION
$$\int \frac{dx}{(\sqrt{2x+3} + \sqrt{2x-3})}$$
$$= \int \frac{1}{(\sqrt{2x+3} + \sqrt{2x-3})} \times \frac{(\sqrt{2x+3}) - \sqrt{2x-3})}{(\sqrt{2x+3}) - \sqrt{2x-3})}\,dx$$
$$= \int \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{[(2x+3) - (2x-3)]}\,dx = \frac{1}{6}\int (2x+3)^{1/2}dx - \frac{1}{6}\int (2x-3)^{1/2}dx$$
$$= \frac{1}{18}(2x+3)^{3/2} - \frac{1}{18}(2x-3)^{3/2} + C.$$

EXAMPLE 15 *Evaluate:*

(i) $\int \frac{1}{(1+\tan x)}\,dx$ *(ii)* $\int \frac{1}{(1+\cot x)}\,dx$

(iii) $\int \left(\frac{1-\tan x}{1+\tan x}\right)dx$ **[CBSE 2000C]** *(iv)* $\int \frac{\tan x}{(\sec x + \cos x)}\,dx$

SOLUTION (i) $\int \frac{1}{(1+\tan x)}\,dx = \int \frac{1}{\left(1 + \frac{\sin x}{\cos x}\right)}\,dx$

$$= \int \frac{\cos x}{(\cos x + \sin x)}\,dx = \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2(\cos x + \sin x)}\,dx$$
$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}\,dx$$
$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{t}\,dt, \text{ where } (\cos x + \sin x) = t \text{ and } (\cos x - \sin x)\,dx = dt$$
$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C = \frac{1}{2}x + \frac{1}{2}\log|\cos x + \sin x| + C.$$

(ii) $\int \frac{1}{(1+\cot x)}\,dx = \int \frac{1}{\left(1+\frac{\cos x}{\sin x}\right)}\,dx = \int \frac{\sin x}{(\sin x+\cos x)}\,dx$

$= \int \frac{(\sin x+\cos x)-(\cos x-\sin x)}{2(\sin x+\cos x)}\,dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\sin x+\cos x)}\,dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{1}{t}\,dt,$

where $\sin x+\cos x = t$ and $(\cos x-\sin x)\,dx = dt$

$= \frac{1}{2}x - \frac{1}{2}\log|t| + C = \frac{1}{2}x - \frac{1}{2}\log|\sin x+\cos x| + C.$

(iii) $\int\left(\frac{1-\tan x}{1+\tan x}\right)dx = \int \frac{\left(1-\frac{\sin x}{\cos x}\right)}{\left(1+\frac{\sin x}{\cos x}\right)}\,dx = \int \frac{(\cos x-\sin x)}{(\cos x+\sin x}\,dx$

$= \int \frac{1}{t}\,dt,$ where $(\cos x+\sin x) = t$ and $(\cos x-\sin x)\,dx = dt$

$= \log|t| + C = \log|(\cos x+\sin x)| + C.$

(iv) $\int \frac{\tan x}{(\sec x+\cos x)}\,dx = \int \frac{\sin x}{1+\cos^2 x}\,dx$

$= -\int \frac{1}{(1+t^2)}\,dt,$ where $\cos x = t$ and $\sin x\,dx = -dt$

$= -\tan^{-1}t + C = -\tan^{-1}(\cos x) + C.$

EXAMPLE 16 *Evaluate:* (i) $\int \tan x\,dx$ (ii) $\int \cot x\,dx$

(iii) $\int \sec x\,dx$ (iv) $\int \operatorname{cosec} x\,dx$

SOLUTION (i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= -\int \frac{1}{t}\,dt,$ where $\cos x = t$ and $\sin x\,dx = -dt$

$= -\log|t| + C = -\log|\cos x| + C.$

$\therefore \int \tan x\,dx = -\log|\cos x| + C.$

(ii) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$= \int \frac{1}{t}\,dt,$ where $\sin x = t$ and $\cos x\,dx = dt$

$= \log|t| + C = \log|\sin x| + C.$

$\therefore \int \cot x\,dx = \log|\sin x| + C.$

(iii) $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$

[multiplying numerator and denominator by $(\sec x + \tan x)$]

$$= \int \frac{1}{t}\,dt, \quad \text{where } (\sec x + \tan x) = t$$

$$\text{and } \sec x(\sec x + \tan x)\,dx = dt$$

$$= \log|t| + C = \log|(\sec x + \tan x)| + C.$$

$\therefore \quad \int \sec x\,dx = \log|(\sec x + \tan x)| + C.$

Alternative form

$$\sec x + \tan x = \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \frac{(1 + \sin x)}{\cos x}$$

Putting $\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$ and $\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$.

$\therefore \quad \sec x + \tan x = \frac{1 + \tan(x/2)}{1 - \tan(x/2)} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$.

$\therefore \quad \int \sec x\,dx = \log|\sec x + \tan x| + C$

$$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C.$$

(iv) $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}\,dx$

[multiplying numerator and denominator by $(\operatorname{cosec} x - \cot x)$]

$$= \int \frac{1}{t}\,dt, \quad \text{where } (\operatorname{cosec} x - \cot x) = t$$

$$\text{and } \operatorname{cosec} x\,(\operatorname{cosec} x - \cot x)\,dx = dt$$

$$= \log|t| + C = \log|\operatorname{cosec} x - \cot x| + C.$$

$\therefore \quad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C.$

Alternative form

$$\operatorname{cosec} x - \cot x = \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \frac{1 - \cos x}{\sin x}$$

$$= \frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} = \tan\frac{x}{2}.$$

$\therefore \quad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

$$= \log|\tan\frac{x}{2}| + C.$$

As a consequence of the above results, the integral of trigonometric functions may be listed as given below:

(i) $\int \sin x\,dx = -\cos x + C$ (ii) $\int \cos x\,dx = \sin x + C$

(iii) $\int \tan x\,dx = -\log|\cos x| + C$

(iv) $\int \cot x\,dx = \log|\sin x| + C$

(v) $\int \sec x\,dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(vi) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

EXAMPLE 17 *Evaluate:*

(i) $\int \frac{(1+\cos x)}{(1-\cos x)}\,dx$ **[CBSE 2000C]** *(ii)* $\int \frac{(1+\sin x)}{(1+\cos x)}\,dx$

SOLUTION (i) $\int \frac{(1+\cos x)}{(1-\cos x)}\,dx = \int \frac{2\cos^2(x/2)}{2\sin^2(x/2)}\,dx$

$= \int \cot^2\left(\frac{x}{2}\right)dx = \int (\operatorname{cosec}^2\frac{x}{2} - 1)\,dx$

$= \int \operatorname{cosec}^2\frac{x}{2}\,dx - \int dx$

$= 2\int \operatorname{cosec}^2 t\,dt - \int dx$, where $\frac{x}{2} = t$ and $dx = 2dt$

$= -2\cot t - x + C = -2\cot\left(\frac{x}{2}\right) - x + C.$

(ii) $\int\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int \frac{1}{(1+\cos x)}dx + \int \frac{\sin x}{(1+\cos x)}dx$

$= \int \frac{1}{2\cos^2(x/2)}dx + \int \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}dx$

$= \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right)dx + \int \tan\frac{x}{2}dx$

$= \int \sec^2 t\,dt + 2\int \tan t\,dt$, where $\frac{x}{2} = t$

$= \tan t - 2\log|\cos t| + C$

$= \tan\left(\frac{x}{2}\right) - 2\log\left|\cos\left(\frac{x}{2}\right)\right| + C.$

EXAMPLE 18 *Evaluate:*

$$(i)\ \int \frac{dx}{1+\sqrt{x}} \qquad (ii)\ \int \frac{x+\sqrt{x+1}}{x+2}\,dx$$

SOLUTION (i) Put $\sqrt{x}=t$ so that $x=t^2$ and $dx=2t\,dt$.

$$\therefore \int \frac{dx}{1+\sqrt{x}} = \int \frac{2t}{(1+t)}\,dt = \int \frac{2(1+t)-2}{(1+t)}\,dt$$

$$= 2\int dt - 2\int \frac{dt}{1+t} = 2t - 2\log|1+t| + C$$

$$= 2\sqrt{x} - 2\log|1+\sqrt{x}| + C.$$

(ii) Put $\sqrt{x+1}=t$ so that $x+1=t^2$ and $dx=2t\,dt$.

$$\therefore \int \frac{x+\sqrt{x+1}}{(x+2)}dx = 2\int \frac{(t^2-1+t)t}{(t^2+1)}dt$$

$$= 2\int \left(\frac{t^3+t^2-t}{t^2+1}\right)dt$$

$$= 2\int \left(t+1-\frac{2t+1}{t^2+1}\right)dt \qquad \text{[by division]}$$

$$= 2\int \left(t+1-\frac{2t}{t^2+1}-\frac{1}{t^2+1}\right)dt$$

$$= 2\int t\,dt + 2\int dt - 2\int \frac{2t}{t^2+1}dt - 2\int \frac{1}{t^2+1}dt$$

$$= t^2 + 2t - 2\log|t^2+1| - 2\tan^{-1}t + C$$

$$= (x+1) + 2\sqrt{x+1} - 2\log|x+2| - 2\tan^{-1}\sqrt{x+1} + C.$$

EXAMPLE 19 *Evaluate* $\int \sqrt{\frac{1+x}{1-x}}\,dx.$ **[CBSE 2006]**

SOLUTION $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$

$$= \int \frac{1+x}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int \frac{1}{\sqrt{t}}dt, \text{ where } (1-x^2)=t$$

$$= \sin^{-1}x - \sqrt{t} + C$$

$$= \sin^{-1}x - \sqrt{1-x^2} + C.$$

EXAMPLE 20 *Evaluate* $\int \frac{(3\sin x - 2)\cos x}{(5-\cos^2 x - 4\sin x)}dx$ **[CBSE 2013C]**

SOLUTION We have

$$I = \int \frac{(3\sin x - 2)\cos x}{\{5 - (1 - \sin^2 x) - 4\sin x\}} dx$$

$$= \int \frac{(3\sin x - 2)\cos x}{\{4 + \sin^2 x - 4\sin x\}} dx = \int \frac{(3\sin x - 2)\cos x}{(2 - \sin x)^2} dx \quad \ldots \text{(i)}$$

Putting $2 - \sin x = t$, we get $\sin x = 2 - t$ and $\cos x\, dx = -dt$.

$$\therefore \quad I = -\int \frac{\{3(2-t) - 2\}}{t^2} dt = -\int \frac{(4 - 3t)}{t^2} dt = \int \frac{(3t - 4)}{t^2} dt$$

$$= \int \left(\frac{3}{t} - \frac{4}{t^2} \right) dt = 3\log |t| + \frac{4}{t} + C$$

$$= 3\log |(2 - \sin x)| + \frac{4}{(2 - \sin x)} + C.$$

EXERCISE 13A

Evaluate the following integrals:

Very-Short-Answer Questions

1. $\int (2x + 9)^5 dx$
2. $\int (7 - 3x)^4 dx$
3. $\int \sqrt{3x - 5}\, dx$
4. $\int \frac{1}{\sqrt{4x + 3}} dx$
5. $\int \frac{1}{\sqrt{3 - 4x}} dx$
6. $\int \frac{1}{(2x - 3)^{3/2}} dx$
7. $\int e^{(2x-1)} dx$
8. $\int e^{(1-3x)} dx$
9. $\int 3^{(2-3x)} dx$
10. $\int \sin 3x\, dx$

Short-Answer Questions

11. $\int \cos(5 + 6x)\, dx$
12. $\int \sin x \sqrt{1 + \cos 2x}\, dx$ **[CBSE 2006C]**
13. $\int \text{cosec}^2(2x + 5)\, dx$
14. $\int \sin x \cos x\, dx$
15. $\int \sin^3 x \cos x\, dx$
16. $\int (\sqrt{\cos x}) \sin x\, dx$
17. $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx$
18. $\int \frac{\sin(2\tan^{-1} x)}{(1 + x^2)} dx$ **[CBSE 2002]**
19. $\int \frac{\cos(\log x)}{x} dx$
20. $\int \frac{\text{cosec}^2(\log x)}{x} dx$ **[CBSE 2001]**
21. $\int \frac{1}{x \log x} dx$
22. $\int \frac{(x + 1)(x + \log x)^2}{x} dx$ **[CBSE 2002]**
23. $\int \frac{(\log x)^2}{x} dx$
24. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

25. $\int e^{\tan x}\sec^2 x\,dx$

26. $\int e^{\cos^2 x}\sin 2x\,dx$

27. $\int \sin(ax+b)\cos(ax+b)\,dx$

28. $\int \cos^3 x\,dx$

29. $\int \frac{1}{x^2}e^{-1/x}\,dx$

30. $\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$

31. $\int \frac{dx}{(e^x+e^{-x})}$

32. $\int \frac{e^{2x}}{(e^{2x}-2)}dx$

33. $\int \cot x \log(\sin x)\,dx$

34. $\int \frac{\cot x}{\log(\sin x)}dx$

35. $\int 2x\sin(x^2+1)\,dx$

36. $\int \sec x \log(\sec x+\tan x)\,dx$

37. $\int \frac{\tan\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx$

38. $\int \frac{x\tan^{-1}x^2}{(1+x^4)}dx$

39. $\int \frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}dx$

40. $\int \frac{1}{(\sqrt{1-x^2})\sin^{-1}x}dx$

41. $\int \frac{\sqrt{(2+\log x)}}{x}dx$

42. $\int \frac{\sec^2 x}{(1+\tan x)}dx$

43. $\int \frac{\sin x}{(1+\cos x)}dx$

44. $\int\left(\frac{1+\tan x}{1-\tan x}\right)dx$ **[CBSE 2000C]**

45. (i) $\int \frac{(1+\tan x)}{(x+\log\sec x)}dx$ (ii) $\int \frac{(1-\sin 2x)}{(x+\cos^2 x)}dx$ **[CBSE 2000]**

46. $\int \frac{\sin 2x}{(a^2+b^2\sin^2 x)}dx$

47. $\int \frac{\sin 2x}{(a^2\cos^2 x+b^2\sin^2 x)}dx$ **[CBSE 2005]**

48. $\int\left(\frac{2\cos x-3\sin x}{3\cos x+2\sin x}\right)dx$

49. $\int \frac{4x}{(2x^2+3)}dx$

50. $\int \frac{(x+1)}{(x^2+2x-3)}dx$

51. $\int \frac{(4x-5)}{(2x^2-5x+1)}dx$

52. $\int \frac{(9x^2-4x+5)}{(3x^3-2x^2+5x+1)}dx$

53. $\int \frac{\sec x\,\mathrm{cosec}\,x}{\log(\tan x)}dx$

54. $\int \frac{(1+\cos x)}{(x+\sin x)^3}dx$

55. $\int \frac{\sin x}{(1+\cos x)^2}dx$

56. $\int \frac{(2x+3)}{\sqrt{x^2+3x-2}}dx$

57. $\int \frac{(2x-1)}{\sqrt{x^2-x-1}}dx$

58. $\int \frac{dx}{(\sqrt{x+a}+\sqrt{x+b})}$

59. $\int \frac{dx}{(\sqrt{1-3x}-\sqrt{5-3x})}$

60. $\int \frac{x^2}{(1+x^6)}dx$

61. $\int \frac{x^3}{(1+x^8)}dx$

62. $\int \frac{x}{(1+x^4)}dx$

63. $\int \frac{x^5}{\sqrt{1+x^3}}dx$

64. $\int \frac{x}{\sqrt{1+x}}dx$

65. $\int \frac{1}{x\sqrt{x^4-1}}dx$

66. $\int x\sqrt{x-1}\,dx$

67. $\int (1-x)\sqrt{1+x}\,dx$

68. $\int x\sqrt{x^2-1}\,dx$

69. $\int x\sqrt{3x-2}\,dx$

70. $\int \frac{dx}{x\cos^2(1+\log x)}$

71. $\int x^2 \sin x^3 dx$

72. $\int (2x+4)\sqrt{x^2+4x+3}\,dx$

73. $\int \frac{\sin x}{(\sin x-\cos x)}dx$

74. $\int \frac{dx}{(1-\tan x)}$

75. $\int \frac{dx}{(1-\cot x)}$

76. $\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$

77. $\int \frac{(\cos x-\sin x)}{(1+\sin 2x)}dx$

78. $\int \frac{(x+1)(x+\log x)^2}{x}dx$

79. $\int x\sin^3 x^2 \cos x^2 dx$

80. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx$

81. $\int e^{-x}\operatorname{cosec}^2(2e^{-x}+5)\,dx$

82. $\int 2x\sec^3(x^2+3)\tan(x^2+3)dx$

83. $\int \frac{\sin 2x}{(a+b\cos x)^2}dx$ **[CBSE 2005]**

84. $\int \frac{dx}{(3-5x)}$

85. $\int \sqrt{1+x}\,dx$

86. $\int x^2 e^{x^3}\cos(e^{x^3})\,dx$

87. $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)}dx$

88. $\int \frac{(x+1)e^x}{\cos^2(xe^x)}dx$

89. $\int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}dx$

90. $\int \sqrt{e^x-1}\,dx$

91. $\int \frac{dx}{(x-\sqrt{x})}$

92. $\int \frac{\sec^2(2\tan^{-1}x)}{(1+x^2)}dx$

93. $\left(\frac{1+\sin 2x}{x+\sin^2 x}\right)dx$ **[CBSE 2000]**

94. $\int\left(\frac{1-\tan x}{x+\log\cos x}\right)dx$ **[CBSE 2000]**

95. $\int \frac{(1+\cot x)}{(x+\log\sin x)}dx$ **[CBSE 2000]**

96. $\int \frac{\tan x\sec^2 x}{(1-\tan^2 x)}dx$

97. $\int \frac{\sin(2\tan^{-1}x)}{(1+x^2)}dx$ **[CBSE 2002]**

98. $\int \frac{dx}{(x^{1/2}+x^{1/3})}$

99. $\int (\sin^{-1}x)^2 dx$ **[CBSE 2004]**

100. $\int \frac{2x\tan^{-1}x^2}{(1+x^4)}dx$ **[CBSE 2006]**

101. $\int \frac{(x^2+1)}{(x^4+1)}dx$ **[CBSE 2006C, '07]**

102. $\int \frac{(\sin x + \cos x)}{\sqrt{\sin 2x}} dx$ **[CBSE 2009C, '10C]**

ANSWERS (EXERCISE 13A)

1. $\frac{(2x+9)^6}{12} + C$ **2.** $\frac{-(7-3x)^5}{15} + C$ **3.** $\frac{2}{9}(3x-5)^{3/2} + C$

4. $\frac{1}{2}\sqrt{4x+3} + C$ **5.** $-\frac{1}{2}\sqrt{3-4x} + C$ **6.** $\frac{-1}{\sqrt{2x-3}} + C$

7. $\frac{1}{2}e^{(2x-1)} + C$ **8.** $-\frac{1}{3}e^{(1-3x)} + C$ **9.** $\frac{-3^{(2-3x)}}{3 \log 3} + C$

10. $\frac{-\cos 3x}{3} + C$ **11.** $\frac{1}{6}\sin(5+6x) + C$ **12.** $\frac{1}{\sqrt{2}}\sin^2 x + C$

13. $-\frac{1}{2}\cot(2x+5) + C$ **14.** $\frac{1}{2}\sin^2 x + C$ **15.** $\frac{1}{4}\sin^4 x + C$

16. $-\frac{2}{3}(\cos x)^{3/2} + C$ **17.** $\frac{1}{2}(\sin^{-1}x)^2 + C$ **18.** $-\frac{1}{2}\cos(2\tan^{-1}x) + C$

19. $\sin(\log x) + C$ **20.** $-\cot(\log x) + C$ **21.** $\log|\log x| + C$

22. $\frac{1}{3}(x + \log x)^3 + C$ **23.** $\frac{1}{3}(\log x)^3 + C$ **24.** $2\sin\sqrt{x} + C$

25. $e^{\tan x} + C$ **26.** $-e^{\cos^2 x} + C$ **27.** $\frac{\sin^2(ax+b)}{2a} + C$

28. $\frac{\sin 3x}{12} + \frac{3}{4}\sin x + C$ **29.** $e^{-1/x} + C$ **30.** $-\sin\left(\frac{1}{x}\right) + C$

31. $\tan^{-1}(e^x) + C$ **32.** $\frac{1}{2}\log|e^{2x} - 2| + C$ **33.** $\frac{1}{2}[\log(\sin x)]^2 + C$

34. $\log|\log(\sin x)| + C$ **35.** $-\cos(x^2+1) + C$

36. $\frac{1}{2}[\log(\sec x + \tan x)]^2 + C$ **37.** $\tan^2\sqrt{x} + C$ **38.** $\frac{1}{4}(\tan^{-1}x^2)^2 + C$

39. $\frac{1}{4}(\sin^{-1}x^2)^2 + C$ **40.** $\log|\sin^{-1}x| + C$ **41.** $\frac{2}{3}(2+\log x)^{3/2} + C$

42. $\log|1+\tan x| + C$ **43.** $-\log|1+\cos x| + C$

44. $-\log|\cos x - \sin x| + C$

45. (i) $\log|x + \log(\sec x)| + C$ (ii) $\log|x + \cos^2 x| + C$

46. $\frac{1}{b^2}\log|a^2 + b^2\sin^2 x| + C$ **47.** $\frac{1}{(b^2-a^2)}\log|a^2\cos^2 x + b^2\sin^2 x| + C$

48. $\log|3\cos x + 2\sin x| + C$ **49.** $\log|2x^2 + 3| + C$

50. $\frac{1}{2}\log|x^2 + 2x - 3| + C$ **51.** $\log|2x^2 - 5x + 1| + C$

52. $\log|3x^3-2x^2+5x+1|+C$ **53.** $\log|\log\tan x|+C$

54. $\dfrac{-1}{2(x+\sin x)^2}+C$ **55.** $\dfrac{1}{(1+\cos x)}+C$ **56.** $2\sqrt{x^2+3x-2}+C$

57. $2\sqrt{x^2-x-1}+C$ **58.** $\dfrac{2}{3(a-b)}[(x+a)^{3/2}-(x+b)^{3/2}]+C$

59. $\dfrac{1}{18}[(1-3x)^{3/2}+(5-3x)^{3/2}]+C$ **60.** $\dfrac{1}{3}\tan^{-1}x^3+C$

61. $\dfrac{1}{4}\tan^{-1}x^4+C$ **62.** $\dfrac{1}{2}\tan^{-1}x^2+C$

63. $\dfrac{2}{9}(1+x^3)^{3/2}-\dfrac{2}{3}(1+x^3)^{1/2}+C$ **64.** $\dfrac{2}{3}(1+x)^{3/2}-2\sqrt{1+x}+C$

65. $\dfrac{1}{2}\sec^{-1}x^2+C$ **66.** $\dfrac{2}{5}(x-1)^{5/2}+\dfrac{2}{3}(x-1)^{3/2}+C$

67. $\dfrac{4}{3}(1+x)^{3/2}-\dfrac{2}{5}(1+x)^{5/2}+C$ **68.** $\dfrac{1}{3}(x^2-1)^{3/2}+C$

69. $\dfrac{2}{45}(3x-2)^{5/2}+\dfrac{4}{27}(3x-2)^{3/2}+C$ **70.** $\tan(1+\log x)+C$

71. $-\dfrac{1}{3}\cos x^3+C$ **72.** $\dfrac{2}{3}(x^2+4x+3)^{3/2}+C$

73. $\dfrac{1}{2}x+\dfrac{1}{2}\log|\sin x-\cos x|+C$ **74.** $\dfrac{1}{2}x-\dfrac{1}{2}\log|\sin x-\cos x|+C$

75. $\dfrac{1}{2}x+\dfrac{1}{2}\log|\sin x-\cos x|+C$ **76.** $\log|\sin x+\cos x|+C$

77. $\dfrac{-1}{(\sin x+\cos x)}+C$ **78.** $\dfrac{1}{3}(x+\log x)^3+C$ **79.** $\dfrac{1}{8}\sin^4x^2+C$

80. $\sin^{-1}(\tan x)+C$ **81.** $\dfrac{1}{2}\cot(2e^{-x}+5)+C$ **82.** $\dfrac{\sec^3(x^2+3)}{3}+C$

83. $\dfrac{-2}{b^2}\left[\log|a+b\cos x|+\dfrac{a}{(a+b\cos x)}\right]+C$ **84.** $\dfrac{-\log|3-5x|}{5}+C$

85. $\dfrac{2}{3}(1+x)^{3/2}+C$ **86.** $\dfrac{1}{3}\sin(e^{x^3})+C$ **87.** $\dfrac{e^{m\tan^{-1}x}}{m}+C$

88. $\tan(xe^x)+C$ **89.** $2\sin(e^{\sqrt{x}})+C$ **90.** $2\sqrt{e^x-1}-2\tan^{-1}\sqrt{e^x-1}+C$

91. $2\log|\sqrt{x}-1|+C$ **92.** $\dfrac{1}{2}\tan(2\tan^{-1}x)+C$ **93.** $\log|x+\sin^2x|+C$

94. $\log|x+\log\cos x|+C$ **95.** $\log|x+\log\sin x|+C$

96. $-\dfrac{1}{2}\log(1-\tan^2x)+C$ **97.** $-\dfrac{1}{2}\cos(2\tan^{-1}x)+C$

98. $2\sqrt{x}-3x^{1/3}+6x^{1/6}-6\log|1+x^{1/6}|+C$

99. $x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$ **100.** $\dfrac{1}{2}(\tan^{-1}x^2)^2+C$

101. $\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right\}+C$ **102.** $\tan^{-1}\left\{\frac{\tan x-1}{\sqrt{2}\tan x}\right\}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 13A)

12. $I=\int \sin x\sqrt{2\cos^2 x}\,dx=\sqrt{2}\int \sin x\cos x\,dx.$ Put $\sin x=t$ and $\cos x\,dx=dt.$

22. $I=\int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx.$ Put $(x+\log x)=t$ and $\left(1+\frac{1}{x}\right)dx=dt$

24. Put $\sqrt{x}=t.$ 25. Put $\tan x=t.$ 26. Put $\cos^2 x=t.$ 27. Put $\sin(ax+b)=t.$

28. $\cos^3 x=\frac{3}{4}\cos x+\frac{1}{4}\cos 3x.$ 29. Put $\frac{-1}{x}=t.$ 32. Put $(e^{2x}-2)=t.$

33. Put $\log(\sin x)=t.$ 35. Put $(x^2+1)=t.$ 36. Put $\log(\sec x+\tan x)=t.$

37. Put $\tan\sqrt{x}=t.$ 38. Put $\tan^{-1}x^2=t.$

39. Put $\sin^{-1}x^2=t.$ 40. Put $\sin^{-1}x=t.$

41. Put $(2+\log x)=t.$ 44. $I=\int\frac{(\cos x+\sin x)}{(\cos x-\sin x)}dx.$ Put $(\cos x-\sin x)=t.$

45. to 53. In each of these questions, put the denominator equal to t.

60. Put $x^3=t.$ 63. Put $(1+x^3)=t^2.$ Then, $I=\frac{2}{3}\int\frac{t(t^2-1)}{\sqrt{t}}dt=\frac{2}{3}\int(t^{5/2}-t^{1/2})dt.$

64. Put $(1+x)=t^2.$ 65. Multiply num. and denom. by x and put $x^2=t.$

73. $I=\int\frac{(\sin x-\cos x)+(\sin x+\cos x)}{2(\sin x-\cos x)}dx.$

74. $I=\int\frac{-\cos x}{(\sin x-\cos x)}dx=\int\frac{(\sin x-\cos x)-(\sin x+\cos x)}{2(\sin x-\cos x)}dx.$

76. $I=\int\frac{(\cos^2 x-\sin^2 x)}{(\sin x+\cos x)^2}dx=\int\frac{(\cos x-\sin x)}{(\sin x+\cos x)}dx.$

77. $I=\int\frac{(\cos x-\sin x)}{(\sin x+\cos x)^2}dx.$ Put $(\sin x+\cos x)=t.$

78. $I=\int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx.$ Put $(x+\log x)=t.$ 79. Put $\sin x^2=t.$

80. Put $\tan x=t.$ 81. Put $(2e^{-x}+5)=t.$ 82. Put $\sec(x^2+3)=t.$

83. Put $(a+b\cos x)=t.$ Then, $\cos x=\frac{(t-a)}{b}$ and $-\sin x\,dx=\frac{1}{b}dt.$

$\therefore\quad I=-\frac{2}{b}\cdot\int\frac{1}{t^2}\left(\frac{t-a}{b}\right)dt=-\frac{2}{b^2}\cdot\int\left\{\frac{1}{t}-\frac{a}{t^2}\right\}dt.$

86. Put $e^{x^3}=t.$ 88. Put $(xe^x)=t.$ 89. Put $e^{\sqrt{x}}=t.$

90. Put $(e^x - 1) = t^2$ and $dx = \dfrac{2t}{(t^2 + 1)}\,dt$. Then,

$$I = \int \frac{2t^2}{(1+t^2)}dt = 2\int\left(1 - \frac{1}{1+t^2}\right)dt.$$

91. $I = \int \dfrac{dx}{\sqrt{x}\,(\sqrt{x}-1)}$. Put $\sqrt{x} - 1 = t$ and $\dfrac{1}{\sqrt{x}}\,dx = 2dt$.

92. Put $2\tan^{-1}x = t$.

96. Put $(1 - \tan^2 x) = t$

97. Put $\tan^{-1}x = t$ and $\dfrac{1}{(1+x^2)}dx = dt$.

98. $I = \int \dfrac{dx}{x^{1/3}(1+x^{1/6})}$. Now put $x = t^6$ and $dx = 6t^5 dt$.

Then, $I = 6\int \dfrac{t^3}{1+t}dt = 6\cdot\int\left(t^2 - t + 1 - \dfrac{1}{1+t}\right)dt$.

101. Divide num. and denom. by x^2. Put $\left(x - \dfrac{1}{x}\right) = t$.

$\therefore\quad I = \int \dfrac{dt}{\{t^2 + (\sqrt{2})^2\}}$.

OBJECTIVE QUESTIONS I

Mark (✓) against the correct answer in each of the following:

1. $\int (2x+3)^5 dx = ?$

(a) $\dfrac{(2x+3)^6}{6} + C$ (b) $\dfrac{(2x+3)^4}{8} + C$ (c) $\dfrac{(2x+3)^6}{12} + C$ (d) none of these

2. $\int (3-5x)^7 dx = ?$

(a) $-5(3-5x)^6 + C$ (b) $\dfrac{(3-5x)^8}{-40} + C$

(c) $\dfrac{-5(3-5x)^8}{8} + C$ (d) none of these

3. $\int \dfrac{1}{(2-3x)^4}dx = ?$

(a) $\dfrac{1}{15(2-3x)^5} + C$ (b) $\dfrac{1}{-12(2-3x)^3} + C$

(c) $\dfrac{1}{9(2-3x)^3} + C$ (d) none of these

4. $\int \sqrt{ax+b}\,dx = ?$

(a) $\dfrac{2(ax+b)^{3/2}}{3a} + C$ (b) $\dfrac{3(ax+b)^{3/2}}{2a} + C$

(c) $\dfrac{1}{2\sqrt{ax+b}} + C$ (d) none of these

5. $\int \sec^2(7-4x)\,dx = ?$

(a) $\frac{1}{4}\tan(7-4x)+C$ (b) $\frac{-1}{4}\tan(7-4x)+C$

(c) $4\tan(7-4x)+C$ (d) $-4\tan(7-4x)+C$

6. $\int \cos 3x\,dx = ?$

(a) $-\frac{1}{3}\sin 3x+C$ (b) $\frac{1}{3}\sin 3x+C$ (c) $3\sin 3x+C$ (d) $-3\sin 3x+C$

7. $\int e^{(5-3x)}dx = ?$

(a) $-3e^{(5-3x)}+C$ (b) $\frac{1}{3}e^{(5-3x)}+C$ (c) $\frac{e^{(5-3x)}}{-3}+C$ (d) none of these

8. $\int 2^{(3x+4)}dx = ?$

(a) $\frac{3}{(\log 2)}\cdot 2^{(3x+4)}+C$ (b) $\frac{2^{(3x+4)}}{3(\log 2)+C}$

(c) $\frac{2^{(3x+4)}}{2(\log 3)}+C$ (d) none of these

9. $\int \tan^2\frac{x}{2}dx = ?$

(a) $\tan\frac{x}{2}-x+C$ (b) $\tan\frac{x}{2}+x+C$

(c) $2\tan\frac{x}{2}+x+C$ (d) $2\tan\frac{x}{2}-x+C$

10. $\int \sqrt{1-\cos x}\,dx = ?$

(a) $-\sqrt{2}\cos\frac{x}{2}+C$ (b) $-2\sqrt{2}\cos\frac{x}{2}+C$ (c) $\frac{-1}{2}\cos\frac{x}{2}+C$ (d) $\frac{-1}{\sqrt{2}}\cos\frac{x}{2}+C$

11. $\int \sqrt{1+\sin x}\,dx = ?$

(a) $-\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)+C$ (b) $\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)+C$

(c) $-2\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)+C$ (d) none of these

12. $\int \sin^3 x\,dx = ?$

(a) $-\frac{3}{4}\cos x+\frac{\cos 3x}{12}+C$ (b) $\frac{3}{4}\cos x+\frac{\cos 3x}{12}+C$

(c) $-\frac{3}{4}\cos x-\frac{\cos 3x}{12}+C$ (d) none of these

13. $\int \frac{\log x}{x} dx = ?$

(a) $\frac{1}{2}(\log x)^2 + C$ (b) $-\frac{1}{2}(\log x)^2 + C$

(c) $\frac{2}{x^2} + C$ (d) $\frac{-2}{x^2} + C$

14. $\int \frac{\sec^2(\log x)}{x} dx = ?$

(a) $\log(\tan x) + C$ (b) $-\log(\tan x) + C$

(c) $\tan(\tan x) + C$ (d) $-\tan(\log x) + C$

15. $\int \frac{1}{x(\log x)} dx = ?$

(a) $\log|x| + C$ (b) $\frac{-2}{x^2} + C$ (c) $(\log x)^2 + C$ (d) $\log|\log x| + C$

16. $\int e^{x^3} x^2 dx = ?$

(a) $e^{x^3} + C$ (b) $\frac{1}{3} e^{x^3} + C$ (c) $\frac{1}{6} e^{x^3} + C$ (d) none of these

17. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = ?$

(a) $e^{\sqrt{x}} + C$ (b) $\frac{1}{2} e^{\sqrt{x}} + C$ (c) $2e^{\sqrt{x}} + C$ (d) none of these

18. $\int \frac{e^{\tan^{-1} x}}{(1+x^2)} dx = ?$

(a) $\frac{e^{\tan^{-1} x}}{x} + C$ (b) $e^{\tan^{-1} x} + C$ (c) $e^x \tan^{-1} x + C$ (d) none of these

19. $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = ?$

(a) $2\cos\sqrt{x} + C$ (b) $-2\cos\sqrt{x} + C$ (c) $-\frac{\cos\sqrt{x}}{2} + C$ (d) $\frac{\cos\sqrt{x}}{2} + C$

20. $\int (\sqrt{\sin x}) \cos x \, dx = ?$

(a) $\frac{2}{3}(\cos x)^{3/2} + C$ (b) $\frac{3}{2}(\cos x)^{3/2} + C$

(c) $\frac{2}{3}(\sin x)^{3/2} + C$ (d) $\frac{3}{2}(\sin x)^{3/2} + C$

21. $\int \frac{1}{(1+x^2)\sqrt{\tan^{-1} x}} dx = ?$

(a) $\frac{1}{2} \log|\tan^{-1} x| + C$ (b) $2\sqrt{\tan^{-1} x} + C$

(c) $\frac{1}{2\sqrt{\tan^{-1} x}} + C$ (d) none of these

22. $\int \frac{\cot x}{\log(\sin x)} dx = ?$

(a) $\log|\cot x| + C$ (b) $\log|\cot x \operatorname{cosec} x| + C$

(c) $\log|\log \sin x| + C$ (d) none of these

23. $\int \frac{1}{x\cos^2(1+\log x)} dx = ?$

(a) $\tan(1+\log x) + C$ (b) $\cot(1+\log x) + C$

(c) $\sec(1+\log x) + C$ (d) none of these

24. $\int \frac{x^2 \tan^{-1} x^3}{(1+x^6)} dx = ?$

(a) $\frac{1}{3}(\tan^{-1} x^3)^2 + C$ (b) $\log|\tan^{-1} x^3| + C$

(c) $\frac{1}{6}(\tan^{-1} x^3)^2 + C$ (d) none of these

25. $\int \sec^5 x \tan x \, dx = ?$

(a) $5\tan^5 x + C$ (b) $\frac{1}{5}\tan^5 x + C$

(c) $5\log|\cos x| + C$ (d) none of these

26. $\int \operatorname{cosec}^3(2x+1)\cot(2x+1) dx = ?$

(a) $\frac{1}{4}\operatorname{cosec}^4(2x+1) + C$ (b) $-\frac{1}{3}\operatorname{cosec}^3(2x+1) + C$

(c) $-\frac{1}{6}\operatorname{cosec}^3(2x+1) + C$ (d) $\frac{1}{2}\operatorname{cosec}(2x+1)\cot(2x+1) + C$

27. $\int \frac{\tan(\sin^{-1} x)}{\sqrt{1-x^2}} dx = ?$

(a) $\log|\sec(\sin^{-1} x)| + C$ (b) $\log|\cos(\sin^{-1} x)| + C$

(c) $\tan(\sin^1 x) + C$ (d) none of these

28. $\int \frac{\tan(\log x)}{x} dx = ?$

(a) $x\tan(\log x) + C$ (b) $\log|\tan x| + C$

(c) $\log|\cos(\log x)| + C$ (d) $-\log|\cos(\log x)| + C$

29. $\int e^x \cot(e^x) dx = ?$

(a) $\cot(e^x) + C$ (b) $\log|\sin e^x| + C$

(c) $\log|\operatorname{cosec} e^x| + C$ (d) none of these

30. $\int \frac{e^x}{\sqrt{1+e^x}}dx = ?$

(a) $2\sqrt{1+e^x}+C$ (b) $\frac{1}{2}\sqrt{1+e^x}+C$

(c) $\frac{1}{\sqrt{1+e^x}}+C$ (d) none of these

31. $\int \frac{x}{\sqrt{1-x^2}}dx = ?$

(a) $\sin^{-1}x+C$ (b) $\sin^{-1}\sqrt{x}+C$ (c) $\sqrt{1-x^2}+C$ (d) $-\sqrt{1-x^2}+C$

32. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx = ?$

(a) $\tan(xe^x)+C$ (b) $\cot(xe^x)+C$ (c) $ex^x \tan x+C$ (d) none of these

33. $\int \frac{dx}{(e^x+e^{-x})} = ?$

(a) $\cot^{-1}(e^x)+C$ (b) $\tan^{-1}(e^x)+C$

(c) $\log|e^x+1|+C$ (d) none of these

34. $\int \frac{2^x}{1-4^x}dx = ?$

(a) $\sin^{-1}(2^x)+C$ (b) $(\log e^2)\sin^{-1}(2^x)+C$

(c) $(\log e^2)\cos^{-1}(2^x)+C$ (d) $(\log_2 e)\sin^{-1}(2^x)+C$

35. $\int \frac{dx}{(e^x-1)} = ?$

(a) $\log|e^x-1|+C$ (b) $\log|1-e^{-x}|+C$

(c) $\log|e^x-1|+C$ (d) none of these

36. $\int \frac{1}{(\sqrt{x}+x)}dx = ?$

(a) $\log|1+\sqrt{x}|+C$ (b) $2\log|1+\sqrt{x}|+C$

(c) $\frac{1}{\sqrt{x}}\tan^{-1}\sqrt{x}+C$ (d) none of these

37. $\int \frac{dx}{(1+\sin x)} = ?$

(a) $\tan x+\sec x+C$ (b) $\tan x-\sec x+C$

(c) $\frac{1}{2}\tan\frac{x}{2}+C$ (d) none of these

38. $\int \frac{\sin x}{(1+\sin x)}dx = ?$

(a) $x+\tan x-\sec x+C$ (b) $x-\tan x-\sec x+C$

(c) $x-\tan x+\sec x+C$ (d) none of these

39. $\int \frac{\sin x}{(1-\sin x)} dx = ?$

(a) $-x + \sec x - \tan x + C$ (b) $x + \cos x - \sin x + C$

(c) $-\log|1-\sin x| + C$ (d) none of these

40. $\int \frac{dx}{(1+\cos x)} = ?$

(a) $\frac{1}{2}\tan\frac{x}{2} + C$ (b) $-\cot\frac{x}{2} + C$

(c) $\tan\frac{x}{2} + C$ (d) none of these

41. $\int \frac{dx}{(1-\cos x)} = ?$

(a) $\frac{1}{(x-\sin x)} + C$ (b) $\log|x-\sin x| + C$

(c) $\log\left|\tan\frac{x}{2}\right| + C$ (d) $-\cot\frac{x}{2} + C$

42. $\int \left\{\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right\} dx = ?$

(a) $2\log\left|\sec\frac{x}{2}\right| + C$ (b) $2\log\left|\cos\frac{x}{2}\right| + C$

(c) $2\log\left|\sec\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C$ (d) $2\log\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C$

43. $\int \sqrt{e^x}\, dx = ?$

(a) $\sqrt{e^x} + C$ (b) $2\sqrt{e^x} + C$ (c) $\frac{1}{2}\sqrt{e^x} + C$ (d) none of these

44. $\int \frac{\cos x}{(1+\cos x)} dx = ?$

(a) $x + \tan\frac{x}{2} + C$ (b) $-x + \tan\frac{x}{2} + C$

(c) $x - \tan\frac{x}{2} + C$ (d) none of these

45. $\int \sec^2 x \operatorname{cosec}^2 x\, dx = ?$

(a) $\tan x - \cot x + C$ (b) $\tan x + \cot x + C$

(c) $-\tan x + \cot x + C$ (d) none of these

46. $\int \frac{(1-\cos 2x)}{(1+\cos 2x)} dx = ?$

(a) $\tan x + x + C$ (b) $\tan x - x + C$ (c) $-\tan x + x + C$ (d) none of these

47. $\int \frac{(1+\cos x)}{(1-\cos x)} dx = ?$

(a) $-2\cot \frac{x}{2} - x + C$ (b) $-2\cot \frac{x}{2} + x + C$

(c) $2\cot \frac{x}{2} + x + C$ (d) none of these

48. $\int \frac{1}{\sin^2 x \cos^2 x} dx = ?$

(a) $\tan x + \cot x + C$ (b) $\tan x - \cot x + C$

(c) $-\tan x + \cot x + C$ (d) none of these

49. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx = ?$

(a) $\cot x + \tan x + C$ (b) $-\cot x + \tan x + C$

(c) $\cot x - \tan x + C$ (d) $-\cot x - \tan x + C$

50. $\int \frac{(\cos 2x - \cos 2\alpha)}{(\cos x - \cos \alpha)} dx = ?$

(a) $\sin x + x \cos \alpha + C$ (b) $2\sin x + x \cos \alpha + C$

(c) $2\sin x + 2x \cos \alpha + C$ (d) none of these

51. $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx = ?$

(a) $2x^2 + C$ (b) $\frac{x^2}{2} + C$ (c) $\frac{2}{(1+x^2)} + C$ (d) none of these

52. $\int \tan^{-1}(\sec x + \tan x)\, dx = ?$

(a) $\frac{\pi x}{4} + \frac{x^2}{4} + C$ (b) $\frac{\pi x}{4} - \frac{x^2}{4} + C$ (c) $\frac{1}{(1+x^2)} + C$ (d) none of these

53. $\int \frac{(1+\sin x)}{(1-\sin x)} dx = ?$

(a) $2\tan x + x - 2\sec x + C$ (b) $2\tan x - x + 2\sec x + C$

(c) $2\tan x - x - 2\sec x + C$ (d) none of these

54. $\int \frac{x^4}{(1+x^2)} dx = ?$

(a) $\frac{x^3}{3} + x + \tan^{-1} x + C$ (b) $\frac{-x^3}{3} + x - \tan^{-1} x + C$

(c) $\frac{x^3}{3} - x + \tan^{-1} x + C$ (d) none of these

55. $\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)}dx = ?$

(a) $x\cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C$

(b) $x\cos 2\alpha + \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C$

(c) $x\cos 2\alpha + \sin \alpha \cdot \log|\sin(x+\alpha)| + C$

(d) none of these

56. $\int \frac{1}{(\sqrt{x+3}-\sqrt{x+2})}dx = ?$

(a) $\frac{2}{3}(x+3)^{3/2} - \frac{2}{3}(x+2)^{3/2} + C$ (b) $\frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x+2)^{3/2} + C$

(c) $\frac{3}{2}(x+3)^{3/2} - \frac{3}{2}(x+2)^{3/2} + C$ (d) none of these

57. $\int \frac{(1+\tan x)}{(1-\tan x)}dx = ?$

(a) $-\log|\cos x - \sin x| + C$ (b) $\log|\cos x - \sin x| + C$

(c) $\log|\cos x + \sin x| + C$ (d) none of these

58. $\int \frac{3x^2}{(1+x^6)}dx = ?$

(a) $\sin^{-1}x^3 + C$ (b) $\cos^{-1}x^3 + C$

(c) $\tan^{-1}x^3 + C$ (d) $\cot^{-1}x^3 + C$

59. $\int \frac{dx}{x\sqrt{x^6-1}} = ?$

(a) $\frac{1}{3}\sec^{-1}x^3 + C$ (b) $\frac{1}{3}\operatorname{cosec}^{-1}x^3 + C$

(c) $\frac{1}{3}\cot^{-1}x^3 + C$ (d) none of these

60. $\int\{(2x+1)\sqrt{x^2+x+1}\}dx = ?$

(a) $\frac{3}{2}(x^2+x+1)^{3/2} + C$ (b) $\frac{2}{3}(x^2+x+1)^{3/2} + C$

(c) $\frac{3}{2}(2x+1)^{3/2} + C$ (d) none of these

61. $\int \frac{dx}{\{\sqrt{2x+3}+\sqrt{2x-3}\}} = ?$

(a) $\frac{1}{18}(2x+3)^{3/2} + \frac{1}{18}(2x-3)^{3/2} + C$ (b) $\frac{1}{18}(2x+3)^{3/2} - \frac{1}{18}(2x-3)^{3/2} + C$

(c) $\frac{1}{12}(2x+3)^{3/2} - \frac{1}{12}(2x-3)^{3/2} + C$ (d) none of these

62. $\int \tan x\,dx = ?$

(a) $\log|\cos x| + C$ (b) $-\log|\cos x| + C$

(c) $\log|\sin x| + C$ (d) $-\log|\sin x| + C$

63. $\int \sec x\,dx = ?$

(a) $\log|\sec x - \tan x| + C$ (b) $-\log|\sec x + \tan x| + C$
(c) $\log|\sec x + \tan x| + C$ (d) none of these

64. $\int \operatorname{cosec} x\,dx = ?$

(a) $\log|\operatorname{cosec} x - \cot x| + C$ (b) $-\log|\operatorname{cosec} x - \cot x| + C$
(c) $\log|\operatorname{cosec} x + \cot x| + C$ (d) none of these

65. $\int \frac{(1+\sin x)}{(1+\cos x)}\,dx = ?$

(a) $\tan\frac{x}{2} + 2\log\left|\cos\frac{x}{2}\right| + C$ (b) $-\tan\frac{x}{2} + 2\log\left|\cos\frac{x}{2}\right| + C$
(c) $\tan\frac{x}{2} - 2\log\left|\cos\frac{x}{2}\right| + C$ (d) none of these

66. $\int \frac{\tan x}{(\sec x + \cos x)}\,dx = ?$

(a) $\tan^{-1}(\cos x) + C$ (b) $-\tan^{-1}(\cos x) + C$
(c) $\cot^{-1}(\cos x) + C$ (d) none of these

67. $\int \sqrt{\frac{1+x}{1-x}}\,dx = ?$

(a) $\sin^{-1}x + \sqrt{1-x^2} + C$ (b) $\sin^{-1}x + (1+x^2) + C$
(c) $\sin^{-1}x - \sqrt{1-x^2} + C$ (d) none of these

68. $\int \frac{1}{x^2}e^{-1/x}\,dx = ?$

(a) $e^{-1/x} + C$ (b) $-e^{-1/x} + C$ (c) $\frac{e^{-1/x}}{x} + C$ (d) none of these

69. $\int \frac{x^3}{(1+x^8)}\,dx = ?$

(a) $\tan^{-1}x^4 + C$ (b) $4\tan^{-1}x^4 + C$
(c) $\frac{1}{4}\tan^{-1}x^4 + C$ (d) none of these

70. $\int \frac{(x+1)(x+\log x)^2}{x}\,dx = ?$

(a) $\frac{1}{3}(x+\log x)^3 + C$ (b) $\frac{x^2}{2} + x + C$
(c) $\frac{x^3}{3} + \frac{x^2}{2} + x + C$ (d) none of these

71. $\int \frac{2x \tan^{-1} x^2}{(1+x^4)} dx = ?$

(a) $(\tan^{-1} x^2)^2 + C$ (b) $2\tan^{-1} x^2 + C$

(c) $\frac{1}{2}(\tan^{-1} x^2)^2 + C$ (d) none of these

72. $\int \frac{dx}{(2-3x)} = ?$

(a) $-3 \log |2 - 3x| + C$ (b) $-\frac{1}{3} \log |2 - 3x| + C$

(c) $-\log |2 - 3x| + C$ (d) none of these

73. $\int x\sqrt{x^2 - 1}\, dx = ?$

(a) $\frac{2}{3}(x^2 - 1)^{3/2} + C$ (b) $\frac{1}{3}(x^2 - 1)^{3/2} + C$

(c) $\frac{1}{\sqrt{x^2 - 1}} + C$ (d) none of these

74. $\int 3^{(5-3x)} dx = ?$

(a) $\frac{-3^{(5-3x)}}{3(\log\ 3)} + C$ (b) $\frac{3^{(4-3x)}}{(\log\ 3)} + C$

(c) $-3^{(5-3x)} \log\ 3 + C$ (d) none of these

75. $\int e^{\tan x} \sec^2 x\, dx = ?$

(a) $e^{\tan x} + \tan\ x + C$ (b) $e^{\tan x} \cdot \tan\ x + C$

(c) $e^{\tan x} + C$ (d) none of these

76. $\int e^{\cos^2 x} \sin\ 2x\, dx = ?$

(a) $e^{\cos^2 x} + C$ (b) $-e^{\cos^2 x} + C$ (c) $e^{\sin^2 x} + C$ (d) none of these

77. $\int x \sin^3 x^2 \cos x^2 dx = ?$

(a) $\frac{1}{4}\sin^4 x^2 + C$ (b) $\frac{1}{8}\sin^4 x^2 + C$ (c) $\frac{1}{2}\sin^4 x^2 + C$ (d) none of these

78. $\int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}})}{\sqrt{x}} dx = ?$

(a) $\sin(e^{\sqrt{x}}) + C$ (b) $\frac{1}{2}\sin(e^{\sqrt{x}}) + C$

(c) $2\sin(e^{\sqrt{x}}) + C$ (d) none of these

79. $\int x^2 \sin\ x^3 dx = ?$

(a) $\cos x^3 + C$ (b) $-\cos x^3 + C$ (c) $-\frac{1}{3}\cos x^3 + C$ (d) none of these

80. $\int \frac{(x+1)e^x}{\cos^2(xe^x)}dx = ?$

(a) $\tan(xe^x) + C$ (b) $-\tan(xe^x) + C$

(c) $\cot(xe^x) + C$ (d) none of these

81. $\int \frac{1}{x\sqrt{x^4-1}}dx = ?$

(a) $\sec^{-1}x^2 + C$ (b) $\frac{1}{2}\sec^{-1}x^2 + C$ (c) $\text{cosec}^{-1}x^2 + C$ (d) none of these

82. $\int x\sqrt{x-1}\,dx = ?$

(a) $\frac{2}{3}(x-1)^{3/2} + C$ (b) $\frac{2}{5}(x-1)^{5/2} + C$

(c) $\frac{2}{5}(x-1)^{5/2} + \frac{3}{2}(x-1)^{3/2} + C$ (d) none of these

83. $\int x\sqrt{x^2-x}\,dx = ?$

(a) $\frac{1}{3}(x^2-1)^{3/2} + C$ (b) $\frac{2}{3}(x^2-1)^{3/2} + C$

(c) $\frac{1}{\sqrt{x^2-1}} + C$ (d) none of these

84. $\int \frac{dx}{(1+\sqrt{x})} = ?$

(a) $\sqrt{x} - \log|1+\sqrt{x}| + C$ (b) $\sqrt{x} + \log|1+\sqrt{x}| + C$

(c) $2\sqrt{x} - 2\log|1+\sqrt{x}| + C$ (d) none of these

85. $\int \sqrt{e^x - 1}\,dx$

(a) $\frac{3}{2}(e^x-1)^{3/2} + C$ (b) $\frac{1}{2}(e^x-1)^{1/2} + C$

(c) $\frac{2}{3}(e^x-1)^{3/2} + C$ (d) none of these

86. $\int \frac{\sin x}{(\sin x - \cos x)}dx = ?$

(a) $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C$ (b) $\frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$

(c) $\log|\sin x - \cos x| + C$ (d) none of these

87. $\int \frac{dx}{(1-\tan x)} = ?$

(a) $\frac{1}{2}\log|\sin x - \cos x| + C$ (b) $\frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$

(c) $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C$ (d) none of these

88. $\int \frac{dx}{(1-\cot x)} = ?$

(a) $\log|\sin x - \cos x| + C$ (b) $\frac{1}{2}\log|\sin x - \cos x| + C$

(c) $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C$ (d) $\frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$

89. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx = ?$

(a) $\sin^{-1}(\tan x) + C$ (b) $\cos^{-1}(\sin x) + C$

(c) $\tan^{-1}(\cos x) + C$ (d) $\tan^{-1}(\sin x) + C$

90. $\int \frac{(x^2+1)}{(x^4+1)} dx = ?$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(x - \frac{1}{x}\right) + C$ (b) $\frac{1}{\sqrt{2}}\cot^{-1}\left\{\left(x - \frac{1}{x}\right)\right\} + C$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x - \frac{1}{x}\right)\right\} + C$ (d) none of these

91. $\int \frac{\sin^6 x}{\cos^8 x} dx = ?$

(a) $\frac{1}{7}\tan^7 x + C$ (b) $\frac{1}{7}\sec^7 x + C$

(c) $\log|\cos^6 x| + C$ (d) none of these

92. $\int \sec^5 x \tan x \, dx = ?$

(a) $\frac{1}{5}\tan^5 x + C$ (b) $\frac{1}{5}\sec^5 x + C$

(c) $5\log|\cos x| + C$ (d) none of these

93. $\int \tan^5 x \, dx = ?$

(a) $\frac{1}{6}\tan^6 x + C$

(b) $\frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + \log|\sec x| + C$

(c) $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log|\sec x| + C$

(d) none of these

94. $\int \sin^3 x \cos^3 x \, dx = ?$

(a) $-\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + C$ (b) $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

(c) $\frac{1}{4}\sin^4 x + \frac{1}{6}\cos^6 x + C$ (d) none of these

95. $\int \sec^4 x \tan x\, dx = ?$

(a) $\frac{1}{2}\sec^2 x + \frac{1}{4}\sec^4 x + C$ (b) $\frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$

(c) $\frac{1}{2}\sec x + \log|\sec x + \tan x| + C$ (d) none of these

96. $\int \frac{\log \tan x}{\sin x \cos x} dx = ?$

(a) $\log\{\log(\tan x)\} + C$ (b) $\frac{1}{2}(\log \tan x)^2 + C$

(c) $\log(\sin x \cos x) + C$ (d) none of these

97. $\int \sin^3(2x+1)\, dx = ?$

(a) $\frac{1}{8}\sin^4(2x+1) + C$

(b) $\frac{1}{2}\cos(2x+1) + \frac{1}{3}\cos^3(2x+1) + C$

(c) $-\frac{1}{2}\cos(2x+1) + \frac{1}{6}\cos^3(2x+1) + C$

(d) none of these

98. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = ?$

(a) $2\sqrt{\tan x} + C$ (b) $2\sqrt{\cot x} + C$ (c) $2\sqrt{\sec x} + C$ (d) none of these

99. $\int \frac{(\cos + \sin x)}{(1 - \sin 2x)} dx = ?$

(a) $\log|\sin x - \cos x| + C$ (b) $\frac{1}{(\cos x - \sin x)} + C$

(c) $\log|\cos x + \sin x| + C$ (d) none of these

100. $\int \sqrt{e^x - 1}\, dx = ?$

(a) $\frac{2}{3}(e^x - 1)^{3/2} + C$ (b) $\frac{1}{2} \cdot \frac{e^x}{\sqrt{e^x - 1}} + C$

(c) $2\sqrt{e^x - 1} - 2\tan^{-1}\sqrt{e^x - 1} + C$ (d) none of these

101. $\int \frac{dx}{\sqrt{\sin^3 x \cos x}} = ?$

(a) $2\sqrt{\tan x} + C$ (b) $2\sqrt{\cot x} + C$ (c) $-2\sqrt{\tan x} + C$ (d) $\frac{-2}{\sqrt{\tan x}} + C$

ANSWERS (OBJECTIVE QUESTIONS I)

1. (c) **2.** (b) **3.** (c) **4.** (a) **5.** (b) **6.** (b) **7.** (c) **8** (b) **9.** (d) **10.** (b)
11. (c) **12.** (a) **13.** (a) **14.** (c) **15.** (d) **16.** (b) **17.** (c) **18.** (b) **19.** (b) **20.** (c)

21. (b) **22.** (c) **23.** (a) **24.** (c) **25.** (b) **26.** (c) **27.** (a) **28.** (d) **29.** (b) **30.** (a)
31. (d) **32.** (a) **33.** (b) **34.** (d) **35.** (b) **36.** (b) **37.** (b) **38.** (c) **39.** (a) **40.** (c)
41. (d) **42.** (d) **43.** (b) **44.** (c) **45.** (a) **46.** (b) **47.** (a) **48.** (b) **49.** (d) **50.** (c)
51. (b) **52.** (a) **53.** (b) **54.** (c) **55.** (a) **56.** (b) **57.** (a) **58.** (c) **59.** (a) **60.** (b)
61. (b) **62.** (b) **63.** (c) **64.** (a) **65.** (c) **66.** (b) **67.** (c) **68.** (a) **69.** (c) **70.** (a)
71. (c) **72.** (b) **73.** (b) **74.** (a) **75.** (c) **76.** (b) **77.** (b) **78.** (c) **79.** (c) **80.** (a)
81. (b) **82.** (c) **83.** (a) **84.** (c) **85.** (d) **86.** (b) **87.** (c) **88.** (d) **89.** (a) **90.** (c)
91. (a) **92.** (b) **93.** (c) **94.** (b) **95.** (b) **96.** (b) **97.** (c) **98.** (a) **99.** (b) **100.** (c)
101. (d)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS I

1. Put $(2x + 3) = t$ and $2dx = dt$.
2. Put $(3 - 5x) = t$ and $-5dx = dt$.
3. Put $(2 - 3x) = t$ and $-3dx = dt$.
4. Put $(ax + b) = t$ and $a\,dx = dt$.

   $\therefore \quad I = \frac{1}{a}\int \sqrt{t}\,dt = \frac{1}{a}\cdot\frac{t^{3/2}}{(3/2)} + C.$
5. Put $(7 - 4x) = t$ and $-4dx = dt$.

   $\therefore \quad I = -\frac{1}{4}\int \sec^2 t\,dt = -\frac{1}{4}\tan t + C.$
6. Put $3x = t$ and $3dx = dt$.
7. Put $(5 - 3x) = t$ and $-3dx = dt$.
8. Put $(3x + 4) = t$ and $3dx = dt$.

   $\therefore \quad I = \frac{1}{3}\int 2^t dt = \frac{1}{3}\cdot\frac{2^t}{\log 2} + C.$
9. $I = \int\left(\sec^2\frac{x}{2} - 1\right)dx = \int \sec^2\frac{x}{2}dx - \int dx.$

   $= 2\int \sec^2 t\,dt - x + C, \quad \text{where } \frac{x}{2} = t$

   $= 2\tan t - x + C = 2\tan\frac{x}{2} + C.$
10. $I = \int \sqrt{2\sin^2(x/2)}\,dx = \sqrt{2}\int \sin(x/2)\,dx.$

    Put $\frac{x}{2} = t$ and $dx = 2dt$.

    $\therefore \quad I = 2\sqrt{2}\int \sin t\,dt = -2\sqrt{2}\cos t + C.$
11. $(1 + \sin x) = 1 + \cos\left(\frac{\pi}{2} - x\right) = 2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right).$

    $\therefore \quad I = \sqrt{2}\int \cos\left(\frac{\pi}{4} - \frac{x}{2}\right)dx.$ Put $\left(\frac{\pi}{4} - \frac{x}{2}\right) = t$ and $\frac{-1}{2}dx = dt.$

    $I = -2\sqrt{2}\int \cos t\,dt = -2\sqrt{2}\sin t + C.$

12. $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.

$\therefore \quad \int \sin^3 x\,dx = \frac{3}{4}\int \sin x\,dx - \frac{1}{4}\int \sin 3x\,dx$

$= \frac{3}{4}(-\cos x) - \frac{1}{4}\frac{(-\cos 3x)}{3} + C = \frac{\cos 3x}{12} - \frac{3}{4}\cos x + C.$

13. Put $\log x = t$ and $\frac{1}{x}dx = dt$.

14. Put $\log x = t$ and $\frac{1}{x}dx = dt$.

15. Put $\log x = t$ and $\frac{1}{x}dx = dt$.

16. Put $x^3 = t$ and $3x^2 dx = dt$.

17. Put $\sqrt{x} = t$ and $\frac{1}{2}x^{-1/2}dx = dt \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$.

18. Put $\tan^{-1}x = t$ and $\frac{1}{(1+x^2)}dx = dt$.

19. Put $\sqrt{x} = t$ and $\frac{1}{\sqrt{x}}dx = 2dt$.

$\therefore \quad I = 2\int \sin dt = -2\cos t + C = -2\cos\sqrt{x} + C.$

20. Put $\sin x = t$ and $\cos x\,dx = dt$.

$\therefore \quad I = \int \sqrt{t}\,dt = \frac{t^{3/2}}{(3/2)} + C = \frac{2}{3}(\sin x)^{3/2} + C.$

21. Put $\tan^{-1}x = t$ and $\frac{1}{(1+x^2)}dx = dt$.

$\therefore \quad I = \int \frac{1}{\sqrt{t}}dt = 2\sqrt{t} + C.$

22. Put $\log(\sin x) = t$ and $\cot x\,dx = dt$.

23. Put $(1 + \log x) = t$ and $\frac{1}{x}dx = dt$.

$\therefore \quad I = \int \frac{1}{\cos^2 t}dt = \int \sec^2 t\,dt = \tan t + C.$

24. Put $\tan^{-1}x^3 = t$ and $\frac{3x^2}{(1+x^6)}dx = dt$.

$\therefore \quad I = \frac{1}{3}\int t\,dt = \frac{1}{6}t^2 + C = \frac{1}{6}(\tan^{-1}x^3)^2 + C.$

25. Put $\sec x = t$ and $\sec x \tan x\,dx = dt$.

$\therefore \quad I = \int \sec^4 x(\sec x \tan x)\,dx = \int t^4 dt = \frac{t^5}{5} + C.$

26. Put $(2x + 1) = t$ and $dx = \frac{1}{2}dt$.

$\therefore \quad I = \frac{1}{2}\int \operatorname{cosec}^2 t(\operatorname{cosec} t \cot t)\,dt$

$$= -\frac{1}{2}\int u^2 du, \text{ where } \operatorname{cosec} t = u$$

$$= -\frac{1}{6}u^3 + C = -\frac{1}{6}\operatorname{cosec}^3 t + C = \frac{-1}{6}\operatorname{cosec}^3(2x+1) + C.$$

27. Put $\sin^{-1} x = t$ and $\frac{1}{\sqrt{1-x^2}}\,dx = dt$.

$\therefore \quad I = \int \tan t\, dt = \log|\sec t| + C.$

28. Put $\log x = t$ and $\frac{1}{x}\,dx = dt$.

$\therefore \quad I = \int \tan t\, dt = -\log|\cos t| + C$

$= -\log|\cos(\log x)| + C.$

29. Put $e^x = t$ and $e^x\, dx = dt$.

$\therefore \quad I = \int \cot t\, dt = \log|\sin t| + C.$

30. Put $(1 + e^x) = t$ and $e^x\, dx = dt$.

$\therefore \quad I = \int \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t} + C.$

31. Put $(1 - x^2) = t$ and $-2x\, dx = dt$.

$\therefore \quad I = -\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = -\frac{1}{2}\int t^{-1/2}\,dt = -\frac{1}{2}\cdot\frac{\sqrt{t}}{(1/2)} + C = -\sqrt{t} + C.$

32. Put $xe^x = t$ and $(xe^x + e^x)\,dx = dt$.

$\therefore \quad I = \int \frac{1}{\cos^2 t}\,dt = \int \sec^2 t\, dt = \tan t + C.$

33. $I = \int \frac{dx}{\left(e^x + \frac{1}{e^x}\right)} = \int \frac{e^x}{(1+e^{2x})}\,dx.$

$= \int \frac{1}{(1+t^2)}\,dt$, where $e^x = t$ and $e^x\, dx = dt$.

$= \tan^{-1}(t) + C = \tan^{-1}(e^x) + C.$

34. Put $2^x = t$ and $2^x(\log 2)\,dx + dt$.

$\therefore \quad I = \int \frac{1}{(\log 2)}\cdot\frac{1}{\sqrt{1-t^2}}\,dt = \frac{1}{(\log_e 2)}\sin^{-1}(t) + C$

$= (\log_2 e)\sin^{-1}(2^x) + C.$

35. $I = \int \frac{dx}{e^x(e^x(1-e^{-x}))} = \int \frac{e^{-x}}{(1-e^{-x})}\,dx.$

Put $(1 - e^{-x}) = t$ and $e^{-x}dx = dt$

$\therefore \quad I = \int \frac{1}{t}\,dt = \log|t| + C + \log|1 - e^{-x}| + C.$

36. $I = \int \frac{dx}{\sqrt{x}(1+\sqrt{x})}\cdot$ Put $(1 + \sqrt{x}) = t$ and $\frac{1}{2\sqrt{x}}\,dx = dt.$

$\therefore \quad I = 2\cdot\int \frac{1}{t}\,dt = 2\log|t| + C.$

37. $I = \int \left\{ \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} \right\} dx = \int \frac{(1-\sin x)}{(1-\sin^2 x)} dx.$

$= \int \frac{(1-\sin x)}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$

$= \int (\sec^2 x - \sec x \tan x)\, dx = \tan x - \sec x + C.$

38. $I = \int \left\{ \frac{\sin x}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} \right\} dx = \int \frac{\sin x(1-\sin x)}{\cos^2 x} dx.$

$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx = \int (\sec x \tan x - \tan^2 x)\, dx$

$= \int \sec x \tan x\, dx - \int (\sec^2 x - 1)\, dx = \sec x - \tan x + x + C.$

39. $I = \int \left\{ \frac{\sin x}{(1-\sin x)} \times \frac{(1+\sin x)}{(1+\sin x)} \right\} dx = \int \left(\frac{\sin x + \sin^2 x}{\cos^2 x} \right) dx.$

$= \int \left\{ \frac{\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \right\} dx = \int (\sec x \tan x + \tan^2 x)\, dx$

$= \int (\sec x \tan x + \sec^2 x - 1)\, dx = \sec x + \tan x - x + C.$

40. $I = \int \frac{dx}{2\cos^2(x/2)} = \frac{1}{2} \int \sec^2 \left(\frac{x}{2} \right) dx.$

$= \int \sec^2 t\, dt,$ where $\frac{x}{2} = t$ and $dx = 2dt$

$= \tan t + C = \tan \frac{x}{2} + C.$

41. $I = \int \frac{dx}{2\sin^2(x/2)} = \frac{1}{2} \int \operatorname{cosec}^2 \left(\frac{x}{2} \right) dx.$

$= \int \operatorname{cosec}^2 t\, dt,$ where $\frac{x}{2} = t$ and $dx = 2dt$

$= -\cot t + C = -\cot (x/2) + C.$

42. $I = \int \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) dx.$ Put $\frac{\pi}{4} - \frac{x}{2} = t$ and $-\frac{1}{2} dx = dt.$

$\therefore \quad I = -2 \int \tan t\, dt = 2 \log |\cos t| + C = 2 \log \left| \cos \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + C.$

43. Put $e^x = t$ and $e^x dx = dt \Rightarrow dx = \frac{1}{t} dt.$

$\therefore \quad I = \int \frac{\sqrt{t}}{t} dt = \int \frac{1}{\sqrt{t}} dt = \int t^{-1/2} dt = \frac{t^{1/2}}{(1/2)} + C = 2\sqrt{t} + C = 2\sqrt{e^x} + C.$

44. $I = \int \left\{ \frac{(1+\cos x) - 1}{(1+\cos x)} \right\} = \int \left\{ 1 - \frac{1}{(1+\cos x)} \right\} dx.$

$= \int dx - \int \frac{dx}{(1+\cos x)} = x - \int \frac{dx}{2\cos^2(x/2)} = x - \frac{1}{2} \int \sec^2 \frac{x}{2}\, dx$

$= x - \frac{1}{2} \times 2 \times \int \sec^2 t \, dt$, where $\frac{x}{2} = t$

$= x - \tan t + C = x - \tan \frac{x}{2} + C.$

45. $I = \int \sec^2 x (1 + \cot^2 x) \, dx = \int \sec^2 x \, dx + \int \sec^2 x \cot^2 x \, dx.$

$= \tan x + \int \frac{1}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x} dx = \tan x + \int \operatorname{cosec}^2 x \, dx$

$= \tan x - \cot x + C.$

46. $I = \int \frac{2 \sin^2 x}{2 \cos^2 x} dx = \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$

47. $I = \frac{2 \cos^2 (x/2)}{2 \sin^2 (x/2)} dx = \int \cot^2 \frac{x}{2} dx = \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1 \right) dx.$

$= \int \operatorname{cosec}^2 \frac{x}{2} dx - \int dx = 2 \int \operatorname{cosec}^2 t \, dt - x + C$

$= -2 \cot t - x + C = -2 \cot \frac{x}{2} - x + C.$

48. $I = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx.$

$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx = \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx = \tan x - \cot x + C.$

49. $I = \int \frac{(\cos^2 x - \sin^2 x)}{\cos^2 x \sin^2 x} dx = \int \frac{\cos^2 x}{\cos^2 x \sin^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x \sin^2 x} dx.$

$= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = \int \operatorname{cosec}^2 x \, dx - \int \sec^2 x \, dx$

$= -\cot x - \tan x + C.$

50. $I = \int \frac{(2 \cos^2 x - 1) - (2 \cos^2 \alpha - 1)}{(\cos x - \cos \alpha)} dx = 2 \int \frac{(\cos^2 x - \cos^2 \alpha)}{(\cos x - \cos \alpha)} dx.$

$= 2 \int (\cos x + \cos \alpha) \, dx = \int 2 \cos x \, dx + 2 \cos \alpha \cdot \int dx$

$= 2 \sin x + 2x \cos \alpha + C.$

51. $I = \int \tan^{-1} \sqrt{\frac{2 \sin^2 x}{2 \cos^2 x}} dx = \int \tan^{-1} (\tan x) \, dx.$

$= \int x \, dx = \frac{x^2}{2} + C.$

52. $\tan^{-1} (\sec x + \tan x) = \tan^{-1} \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x} \right) = \tan^{-1} \left\{ \frac{1 + \sin x}{\cos x} \right\}.$

$= \tan^{-1} \left\{ \frac{1 - \cos \left(\frac{\pi}{2} + x \right)}{\sin \left(\frac{\pi}{2} + x \right)} \right\} = \tan^{-1} \left\{ \frac{2 \sin^2 \left(\frac{\pi}{4} + \frac{x}{2} \right)}{2 \sin \left(\frac{\pi}{4} + \frac{x}{2} \right) \cos \left(\frac{\pi}{4} + \frac{x}{2} \right)} \right\}$

$= \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right\} = \left(\frac{\pi}{4} + \frac{x}{2} \right)$

$$\therefore \qquad I = \int \left(\frac{\pi}{4} + \frac{x}{2}\right) dx = \frac{\pi x}{4} + \frac{x^2}{4} + C.$$

53. $$I = \int \frac{(1+\sin x)}{(1-\sin x)} \times \frac{(1+\sin x)}{(1+\sin x)}\, dx.$$

$$= \int \frac{(1+\sin x)^2}{(1-\sin^2 x)}\, dx = \int \frac{1+\sin^2 x + 2\sin x}{\cos^2 x}\, dx$$

$$= \int \left\{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos^2 x}\right\} dx$$

$$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x)\, dx$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x)\, dx = 2\tan x - x + 2\sec x + C.$$

54. On dividing x^4 by $(x^2 + 1)$, we get:

$$I = \int \left\{(x^2 - 1) + \frac{1}{(x^2+1)}\right\} dx = \frac{x^3}{3} - x + \tan^{-1} x + C.$$

55. $$\frac{\sin(x-\alpha)}{\sin(x+\alpha)} = \frac{\sin(x+\alpha-2\alpha)}{\sin(x+\alpha)} = \frac{\sin(x+\alpha)\cos 2\alpha - \cos(x+\alpha)\sin 2\alpha}{\sin(x+\alpha)}.$$

$$= \cos 2\alpha - \sin 2\alpha \cot(x+\alpha)$$

$$\therefore \qquad I = \int \{\cos 2\alpha - \sin 2\alpha \cdot \cot(x+\alpha)\}\, dx$$

$$= (\cos 2\alpha) \cdot \int dx - \sin 2\alpha \cdot \int \cot(x+\alpha)\, dx$$

$$= x\cos 2\alpha - \sin 2\alpha \cdot \log|\sin(x+\alpha)| + C.$$

56. $$I = \int \frac{1}{\{\sqrt{x+3} - \sqrt{x+2}\}} \times \frac{\{\sqrt{x+3} + \sqrt{x+2}\}}{\{\sqrt{x+3} + \sqrt{x+2}\}}\, dx.$$

$$= \int \left\{\frac{\sqrt{x+3} + \sqrt{x+2}}{(x+3) - (x+2)}\right\} dx = \int \sqrt{x+3}\, dx + \int \sqrt{x+2}\, dx$$

$$= \frac{(x+3)^{3/2}}{(3/2)} + \frac{(x+2)^{3/2}}{(3/2)} + C = \frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x+2)^{3/2} + C.$$

57. $$I = \int \frac{\left(1 + \frac{\sin x}{\cos x}\right)}{\left(1 - \frac{\sin x}{\cos x}\right)} dx = \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx.$$

$$= -\int \frac{dt}{t}, \text{ where } (\cos x - \sin x) = t$$

$$= -\log|t| + C = -\log|\cos x - \sin x| + C.$$

58. Putting $x^3 = t$ and $3x^2 dx = dt$, we get:

$$I = \int \frac{dt}{(1+t^2)} = \tan^{-1} t + C = \tan^{-1} x^3 + C.$$

59. $I = \int \frac{x^2}{x^3\sqrt{x^6-1}}\,dx.$

$= \frac{1}{3}\int \frac{dt}{t\sqrt{t^2-1}}, \quad$ where $x^3 = t$

$= \frac{1}{3}\sec^{-1}t + C = \frac{1}{3}\sec^{-1}x^3 + C.$

60. Put $x^2 + x + 1 = t$ and $(2x + 1)\,dx = dt.$

$I = \int \sqrt{t}\,dt = \frac{t^{3/2}}{(3/2)} + C = \frac{2}{3}(x^2 + x + 1)^{3/2} + C.$

61. $I = \int \frac{1}{\{\sqrt{2x+3} + \sqrt{2x-3}\}} \times \frac{\{\sqrt{2x+3} - \sqrt{2x-3}\}}{\{\sqrt{2x+3} - \sqrt{2x-3}\}}\,dx.$

$= \int \frac{\{\sqrt{2x+3} - \sqrt{2x-3}\}}{\{(2x+3) - (2x-3)\}}\,dx = \frac{1}{6}\int \{\sqrt{2x+3} - \sqrt{2x-3}\}\,dx$

$= \frac{1}{18}(2x+3)^{3/2} - \frac{1}{18}(2x+3)^{3/2} + C.$

62. $I = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-dt}{t}$, where $\cos x = t.$

$= -\log|t| + C = -\log|\cos x| + C.$

63. $I = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{1}{t}\,dt$, where $(\sec x + \tan x) = t.$

$= \log|t| + C = \log|\sec x + \tan x| + C.$

64. $I = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}\,dx = \int \frac{1}{t}\,dt$, where $(\operatorname{cosec} x - \cot x) = t.$

$= \log|t| + C = \log|\operatorname{cosec} x - \cot x| + C.$

65. $I = \int \frac{1}{(1+\cos x)}\,dx + \int \frac{\sin x}{(1+\cos x)}\,dx.$

$= \int \frac{dx}{2\cos^2(x/2)} + \int \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\,dx$

$= \frac{1}{2}\int \sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$

$= \int \sec^2 t\,dt + 2\int \tan t\,dt, \text{ where } \frac{x}{2} = t$

$= \tan t - 2\log|\cos t| + C = \tan\frac{x}{2} - 2\log\left|\cos\frac{x}{2}\right| + C.$

66. $I = \int \frac{\sin x}{(1+\cos^2 x)}\,dx = -\int \frac{1}{(1+t^2)}\,dt$, where $\cos x = t.$

$= -\tan^{-1}t + C = -\tan^{-1}(\cos x) + C.$

67. $I = \int \left\{\frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\right\}dx = \int \frac{(1+x)}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}\,dx = \sin^{-1}x - \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt, \text{ where } (1-x^2) = t$

$= \sin^{-1}x - \sqrt{t} + C = \sin^{-1}x - \sqrt{1-x^2} + C.$

68. Put $-\frac{1}{x} = t$ and $\frac{1}{x^2}\,dx = dt$.

$\therefore \quad I = \int e^t dt = e^t + C = e^{-1/x} + C.$

69. Put $x^4 = t$ and $4x^3\,dx = dt$.

$\therefore \quad I = \frac{1}{4}\int \frac{dt}{(1+t^2)} = \frac{1}{4}\tan^{-1} t + C = \frac{1}{4}\tan^{-1} x^4 + C.$

70. Put $(x + \log x) = t$ and $\left(1 + \frac{1}{x}\right)dx = dt$.

$\therefore \quad I = \int \left(\frac{x+1}{x}\right)\cdot (x + \log x)^2\,dx = \int \left(1 + \frac{1}{x}\right)(x + \log x)^2\,dx$

$= \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}(x + \log x)^3 + C.$

71. Put $\tan^{-1} x^2 = t$ and $\frac{2x}{(1 + x^4)}\,dx = dt$.

$\therefore \quad I = \int t\,dt = \frac{t^2}{2} + C = \frac{1}{2}(\tan^{-1} x^2)^2 + C.$

72. Put $(2 - 3x) = t$ and $-3dx = dt$.

$\therefore \quad I = -\frac{1}{3}\int \frac{1}{t} dt = -\frac{1}{3}\log|t| + C = -\frac{1}{3}\log|2 - 3x| + C.$

73. Put $(x^2 - 1) = t$ and $2x\,dx = dt$.

$\therefore \quad I = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\cdot\frac{t^{3/2}}{(3/2)} + C = \frac{1}{3}t^{3/2} + C = \frac{1}{3}(x^2 - 1)^{3/2} + C.$

74. Put $5 - 3x = t$ and $-3dx = dt$.

$\therefore \quad I = -\frac{1}{3}\int 3^t dt = -\frac{1}{3}\times\frac{3^t}{\log 3} + C = -\frac{1}{3}\times\frac{3^{(5-3x)}}{\log 3} + C.$

75. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

$\therefore \quad I = \int e^t dt = e^t + C = e^{\tan x} + C.$

76. Put $\cos^2 x = t$ and $-2\cos x \sin x\,dx = dt$.

$\therefore \quad I = -\int e^t dt = -e^t + C = -e^{\cos^2 x} + C.$

77. Put $\sin x^2 = t$ and $2x\cos x^2 dx = dt$.

$\therefore \quad I = \frac{1}{2}\int t^3 dt = \frac{1}{2}\times\frac{t^4}{4} + C = \frac{t^4}{8} + C = \frac{1}{8}\sin^4 x^2 + C.$

78. Put $e^{\sqrt{x}} = t$ and $e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\,dx = dt$.

$\therefore \quad I = 2\int \cos t\,dt + C = 2\sin t + C = 2\sin(e^{\sqrt{x}}) + C.$

79. Put $x^3 = t$ and $3x^2 dx = dt$.

$I = \frac{1}{3}\int \sin t\,dt = \frac{1}{3}(-\cos t) + C = -\frac{1}{3}\cos x^3 + C.$

80. Put $xe^x = t$ and $(xe^x + e^x \cdot 1)\,dx = dt$, i.e., $(x+1)e^x\,dx = dt$.

$$\therefore \quad I = \int \frac{1}{\cos^2 t}\,dt = \int \sec^2 t\,dt = \tan t + C = \tan(xe^x) + C.$$

81. $I = \int \frac{x}{x^2\sqrt{x^4-1}}\,dx = \frac{1}{2}\int \frac{dt}{t\sqrt{t^2-1}}$, where $x^2 = t$.

$$= \frac{1}{2}\sec^{-1} t + C = \frac{1}{2}\sec^{-1} x^2 + C.$$

82. $I = \int \{(x-1+1)\sqrt{x-1}\}\,dx = \int \{(x-1)^{3/2} + (x-1)^{1/2}\}\,dx.$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{3}{2}(x-1)^{3/2} + C.$$

83. Put $x^2 - 1 = t$ and $2x\,dx = dt$.

$$I = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\cdot\frac{t^{3/2}}{3/2} + C = \frac{1}{3}t^{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C.$$

84. Put $\sqrt{x} = t$, so that $x = t^2$ and $dx = 2t\,dt$.

$$\therefore \quad I = \int \frac{dx}{(1+\sqrt{x})} = \int \frac{2t}{(1+t)}\,dt = \int \frac{\{2(1+t)-2\}}{(1+t)}\,dt$$

$$= \int \left\{2 - \frac{2}{(1+t)}\right\} dt = 2t - 2\log|1+t| + C$$

$$= 2\sqrt{x} - 2\log|1+\sqrt{x}| + C.$$

85. Put $(e^x - 1) = t^2$. Then $e^x\,dx = 2t\,dt$. So, $dx = \frac{2t}{(t^2+1)}$.

$$\therefore \quad I = \int \frac{2t^2}{(t^2+1)}\,dt = 2\int \frac{\{(t^2+1)-1\}}{(t^2+1)}\,dt = 2\int \left(1 - \frac{1}{t^2+1}\right) dt$$

$$= 2[t - \tan^{-1} t] + C = 2[\sqrt{e^x - 1} - \tan^{-1}\{\sqrt{e^x - 1}\}] + C.$$

86. $I = \int \frac{(\sin x - \cos x) + (\cos x + \sin x)}{2(\sin x - \cos x)}\,dx.$

$$= \int \frac{1}{2}\,dx + \frac{1}{2}\int \frac{(\cos x + \sin x)}{(\sin x - \cos x)}\,dx = \frac{1}{2}x + \frac{1}{2}\int \frac{1}{t}\,dt, \text{ where } t = (\sin x - \cos x)$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C = \frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C.$$

87. $I = \int \frac{\cos x}{(\cos x - \sin x)}\,dx = \int \frac{-\cos x}{(\sin x - \cos x)}\,dx.$

$$= \int \frac{(\sin x - \cos x) - (\sin x + \cos x)}{2(\sin x - \cos x)}\,dx = \int \left\{\frac{1}{2} - \frac{1}{2}\cdot\frac{(\sin x + \cos x)}{(\sin x - \cos x)}\right\} dx$$

$$= \frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C.$$

88. $I = \int \frac{\sin x}{(\sin x - \cos x)}\,dx$, which is the same as Q. 86.

89. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

$$\therefore \quad I = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + C = \sin^{-1}(\tan x) + C.$$

90. On dividing the numerator and denominator by x^2, we get

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left\{\left(x-\frac{1}{x}\right)^2+2\right\}}dx$$

$$= \int \frac{dt}{(t^2+2)}, \text{ where } \left(x-\frac{1}{x}\right) = t \text{ and } \left(1+\frac{1}{x^2}\right)dx = dt$$

$$= \int \frac{dt}{[t^2+(\sqrt{2})^2]} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right\} + C.$$

91. $I = \int \frac{\sin^6 x}{\cos^6 x}\cdot\frac{1}{\cos^2 x}dx = \int \tan^6 x \sec^2 x\,dx = \int t^6 dt$, where $\tan x = t$.

$$= \frac{t^7}{7} + C = \frac{1}{7}\tan^7 x + C.$$

92. $I = \sec^4 x \cdot \sec x \tan x\,dx = \int t^4 dt$, where $\sec x = t$.

$$= \frac{1}{5}t^5 + C = \frac{1}{5}\sec^5 x + C.$$

93. $I = \int \tan^3 x \tan^2 x\,dx = \int \tan^3 x\,(\sec^2 x - 1)\,dx.$

$$= \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx = \int t^3 dt - \int \tan x(\sec^2 x - 1)\,dx, \text{ where } \tan x = t$$

$$= \frac{t^4}{4} - \int \tan x \sec^2 x\,dx + \int \tan x\,dx$$

$$= \frac{1}{4}\tan^4 x - \int u\,du + \log|\sec x| + C, \text{ where } \tan x = u$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}u^2 + \log|\sec x| + C = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log|\sec x| + C.$$

94. $I = \int \sin^3 \cos^2 x \cos x\,dx = \int \sin^3 x(1-\sin^2 x)\cos x\,dx.$

$$= \int t^3(1-t^2)dt, \text{ where } \sin x = t$$

$$= \int (t^3 - t^5)\,dt = \frac{t^4}{4} - \frac{t^6}{6} + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.$$

95. $I = \int \sec^2 x \cdot \sec^2 x \cdot \tan x\,dx.$

$$= \int \sec^2 x(1+\tan^2 x)\tan x\,dx = \int (1+t^2)t\,dt, \text{ where } \tan x = t$$

$$= \int (t + t^3)\,dt = \frac{t^2}{2} + \frac{t^4}{4} + C = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.$$

96. Put $\log \tan x = t$. Then, $\frac{1}{\tan x}\cdot \sec^2 x\,dx = dt$, i.e., $\frac{1}{\sin x \cos x}dx = dt$.

$$\therefore \quad I = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}(\log \tan x)^2 + C.$$

97. Put $(2x+1)=t$ and $dx=\frac{1}{2}dt$.

$$\therefore \quad I=\frac{1}{2}\int \sin^3 t\,dt=\frac{1}{2}\int(1-\cos^2 t)\sin t\,dt$$

$$=\frac{1}{2}\int(1-u^2)(-du)=\frac{1}{2}\int(u^2-1)\,du=\frac{1}{2}\left[\frac{u^3}{3}-u\right]+C, \text{ where } \cos t=u$$

$$=\frac{1}{6}u^3-\frac{1}{2}u+C=\frac{1}{6}\cos^3 t-\frac{1}{2}\cos t+C=\frac{1}{6}\cos^3(2x+1)-\frac{1}{2}\cos(2x+1)+C.$$

98. Dividing Nr and Dr by $\cos^2 x$, we get

$$I=\int\frac{\sqrt{\tan x}}{\tan x}\cdot\sec^2 x\,dx=\int\frac{\sec^2 x}{\sqrt{\tan x}}dx$$

$$=\int\frac{dt}{\sqrt{t}}, \text{ where } \tan x=t$$

$$=2\sqrt{t}+C=2\sqrt{\tan x}+C.$$

99. $I=\int\frac{(\cos x+\sin x)}{(\cos^2 x+\sin^2 x-2\sin x\cos x)}dx$

$$=\int\frac{(\cos x+\sin x)}{(\sin x-\cos x)^2}dx=\int\frac{1}{t^2}dt, \text{ where } (\sin x-\cos x)=t$$

$$=\frac{-1}{t}+C=\frac{-1}{(\sin x-\cos x)}+C=\frac{1}{(\cos x-\sin x)}+C.$$

100. Putting $(e^x-1)=t^2$ and $e^x dx=2t\,dt$.

$$\therefore \quad dx=\frac{2t}{(t^2+1)}dt.$$

$$\therefore \quad I=\int\frac{2t^2}{(t^2+1)}dt=\int\left\{2-\frac{2}{(t^2+1)}\right\}dt=2t-2\tan^{-1}t+C$$

$$=2\sqrt{e^x-1}-2\tan^{-1}\sqrt{e^x-1}+C.$$

101. On dividing Nr and Dr by $\cos^2 x$, we get

$$I=\int\frac{\sec^2 x}{\sqrt{\tan^3 x}}dx=\int\frac{dt}{\sqrt{t^3}}, \text{where } \tan x=t.$$